

forall χ
Cambridge 2014-15
Solutions Booklet

P.D. Magnus
University at Albany, State University of New York

Tim Button
University of Cambridge

This booklet contains model answers to the practice exercises found in `forall χ :Cambridge 2014-15`. For several of the questions, there are multiple correct possible answers; in each case, this booklet contains at most one answer. Answers are given in blue; please contact Tim Button at <http://www.nottub.com/forallx.shtml> if you have accessibility requirements.

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Typesetting was carried out entirely in L^AT_EX2 ϵ . The style for typesetting proofs is based on `fitch.sty` (v0.4) by Peter Selinger, University of Ottawa.

Contents

1 Arguments	1
2 Valid arguments	2
3 Other logical notions	4
5 Connectives	5
6 Sentences of TFL	9
10 Complete truth tables	10
11 Semantic concepts	13
12 Truth table shortcuts	17
13 Partial truth tables	19
15 Sentences with one quantifier	22
16 Multiple generality	26
17 Identity	30
18 Definite descriptions	31
19 Sentences of FOL	34
21 Truth in FOL	35
26 Basic rules for TFL	37
27 Additional rules for TFL	42
28 Proof-theoretic concepts	44
30 Derived rules	51
31 Basic rules for FOL	52
32 Conversion of quantifiers	63
33 Rules for identity	68
34 Derived rules	75

Arguments

1

Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. And Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

Valid arguments

2

A. Which of the following arguments is valid? Which is invalid?

1. Socrates is a man.
2. All men are carrots.

So: Therefore, Socrates is a carrot.

Valid

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.

So: Abe Lincoln was born in Illinois.

Valid

1. If I pull the trigger, Abe Lincoln will die.
2. I do not pull the trigger.

So: Abe Lincoln will not die.

Invalid

Abe Lincoln might die for some other reason: someone else might pull the trigger; he might die of old age.

1. Abe Lincoln was either from France or from Luxemborg.
2. Abe Lincoln was not from Luxemborg.

So: Abe Lincoln was from France.

Valid

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.

So: The world will not end today.

Valid

1. Joe is now 19 years old.
2. Joe is now 87 years old.

So: Bob is now 20 years old.

Valid

An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. It is impossible for all the premises to be true; so it is certainly impossible that the premises are all true and the conclusion is false.

B. Could there be:

1. A valid argument that has one false premise and one true premise? **Yes.**
Example: the first argument, above.
2. A valid argument that has only false premises? **Yes.**
Example: Socrates is a frog, all frogs are excellent pianists, therefore Socrates is an excellent pianist.

-
3. A valid argument with only false premises and a false conclusion? **Yes.**
The same example will suffice.
 4. A sound argument with a false conclusion? **No.**
By definition, a sound argument has true premises. And a valid argument is one where it is impossible for the premises to be true and the conclusion false. So the conclusion of a sound argument is certainly true.
 5. An invalid argument that can be made valid by the addition of a new premise? **Yes.**
Plenty of examples, but let me offer a more general observation. We can *always* make an invalid argument valid, by adding a contradiction into the premises. For an argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. If the premises are contradictory, then it is impossible for them all to be true (and the conclusion false).
 6. A valid argument that can be made invalid by the addition of a new premise? **No.**
An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. Adding another premise will only make it harder for the premises all to be true together.

In each case: if so, give an example; if not, explain why not.

Other logical notions

3

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon. Contingent
2. Someone once crossed the Rubicon. Contingent
3. No one has ever crossed the Rubicon. Contingent
4. If Caesar crossed the Rubicon, then someone has. Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon. Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar. Contingent

B. Look back at the sentences G1–G4 in this section (about giraffes, gorillas and martians in the wild animal park), and consider each of the following:

1. G2, G3, and G4 Jointly consistent
2. G1, G3, and G4 Jointly inconsistent
3. G1, G2, and G4 Jointly consistent
4. G1, G2, and G3 Jointly consistent

Which are jointly consistent? Which are jointly inconsistent?

C. Could there be:

1. A valid argument, the conclusion of which is necessarily false?
Yes: '1 + 1 = 3. So 1 + 2 = 4.'
2. An invalid argument, the conclusion of which is necessarily true?
No. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.
3. Jointly consistent sentences, one of which is necessarily false?
No. If a sentence is necessarily false, there is no way to make it true, let alone it along with all the other sentences.
4. Jointly inconsistent sentences, one of which is necessarily true?
Yes. '1 + 1 = 4' and '1 + 1 = 2'.

In each case: if so, give an example; if not, explain why not.

Connectives

5

A. Using the symbolisation key given, symbolise each English sentence in TFL.

M : Those creatures are men in suits.
 C : Those creatures are chimpanzees.
 G : Those creatures are gorillas.

1. Those creatures are not men in suits.
 $\neg M$
2. Those creatures are men in suits, or they are not.
 $(M \vee \neg M)$
3. Those creatures are either gorillas or chimpanzees.
 $(G \vee C)$
4. Those creatures are neither gorillas nor chimpanzees.
 $\neg(C \vee G)$
5. If those creatures are chimpanzees, then they are neither gorillas nor men in suits.
 $(C \rightarrow \neg(G \vee M))$
6. Unless those creatures are men in suits, they are either chimpanzees or they are gorillas.
 $(M \vee (C \vee G))$

B. Using the symbolisation key given, symbolise each English sentence in TFL.

A : Mister Ace was murdered.
 B : The butler did it.
 C : The cook did it.
 D : The Duchess is lying.
 E : Mister Edge was murdered.
 F : The murder weapon was a frying pan.

1. Either Mister Ace or Mister Edge was murdered.
 $(A \vee E)$
2. If Mister Ace was murdered, then the cook did it.
 $(A \rightarrow C)$
3. If Mister Edge was murdered, then the cook did not do it.
 $(E \rightarrow \neg C)$
4. Either the butler did it, or the Duchess is lying.
 $(B \vee D)$
5. The cook did it only if the Duchess is lying.
 $(C \rightarrow D)$

6. If the murder weapon was a frying pan, then the culprit must have been the cook.
 $(F \rightarrow C)$
7. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
 $(\neg F \rightarrow (C \vee B))$
8. Mister Ace was murdered if and only if Mister Edge was not murdered.
 $(A \leftrightarrow \neg E)$
9. The Duchess is lying, unless it was Mister Edge who was murdered.
 $(D \vee E)$
10. If Mister Ace was murdered, he was done in with a frying pan.
 $(A \rightarrow F)$
11. Since the cook did it, the butler did not.
 $(C \wedge \neg B)$
12. Of course the Duchess is lying!
 D

C. Using the symbolisation key given, symbolise each English sentence in TFL.

- E_1 : Ava is an electrician.
 E_2 : Harrison is an electrician.
 F_1 : Ava is a firefighter.
 F_2 : Harrison is a firefighter.
 S_1 : Ava is satisfied with her career.
 S_2 : Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
 $(E_1 \wedge E_2)$
2. If Ava is a firefighter, then she is satisfied with her career.
 $(F_1 \rightarrow S_1)$
3. Ava is a firefighter, unless she is an electrician.
 $(F_1 \vee E_1)$
4. Harrison is an unsatisfied electrician.
 $(E_2 \wedge \neg S_2)$
5. Neither Ava nor Harrison is an electrician.
 $\neg(E_1 \vee E_2)$
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
 $((E_1 \wedge E_2) \wedge \neg(S_1 \vee S_2))$
7. Harrison is satisfied only if he is a firefighter.
 $(S_2 \rightarrow F_2)$
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
 $((\neg E_1 \rightarrow \neg E_2) \wedge (E_1 \rightarrow E_2))$
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
 $(S_1 \leftrightarrow \neg S_2)$
10. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.
 $((E_2 \wedge F_2) \rightarrow S_2)$

11. It cannot be that Harrison is both an electrician and a firefighter.
 $\neg(E_2 \wedge F_2)$
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
 $((F_2 \wedge F_1) \leftrightarrow \neg(E_2 \vee E_1))$

D. Give a symbolisation key and symbolise the following English sentences in TFL.

A: Alice is a spy.
B: Bob is a spy.
C: The code has been broken.
G: The German embassy will be in an uproar.

1. Alice and Bob are both spies.
 $(A \wedge B)$
2. If either Alice or Bob is a spy, then the code has been broken.
 $((A \vee B) \rightarrow C)$
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
 $(\neg(A \vee B) \rightarrow \neg C)$
4. The German embassy will be in an uproar, unless someone has broken the code.
 $(G \vee C)$
5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
 $((C \vee \neg C) \wedge G)$
6. Either Alice or Bob is a spy, but not both.
 $((A \vee B) \wedge \neg(A \wedge B))$

E. Give a symbolisation key and symbolise the following English sentences in TFL.

F: There is food to be found in the pridelands.
R: Rafiki will talk about squashed bananas.
A: Simba is alive.
K: Scar will remain as king.

1. If there is food to be found in the pridelands, then Rafiki will talk about squashed bananas.
 $(F \rightarrow R)$
2. Rafiki will talk about squashed bananas unless Simba is alive.
 $(R \vee A)$
3. Rafiki will either talk about squashed bananas or he won't, but there is food to be found in the pridelands regardless.
 $((R \vee \neg R) \wedge F)$
4. Scar will remain as king if and only if there is food to be found in the pridelands.
 $(K \leftrightarrow F)$
5. If Simba is alive, then Scar will not remain as king.
 $(A \rightarrow \neg K)$

F. For each argument, write a symbolisation key and symbolise all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

P: Dorothy plays the Piano in the morning.

C: Roger wakes up cranky.

D: Dorothy is distracted.

$(P \rightarrow C), (P \vee D), (\neg C \rightarrow D)$

2. It will either rain or snow on Tuesday. If it rains, Neville will be sad. If it snows, Neville will be cold. Therefore, Neville will either be sad or cold on Tuesday.

*T*₁: It rains on Tuesday

*T*₂: It snows on Tuesday

S: Neville is sad on Tuesday

C: Neville is cold on Tuesday

$(T_1 \vee T_2), (T_1 \rightarrow S), (T_2 \rightarrow C), (S \vee C)$

3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean; but not both.

Z: Zoog remembered to do his chores

C: Things are clean

N: Things are neat

$(Z \rightarrow (C \wedge \neg N)), (\neg Z \rightarrow (N \wedge \neg C)), ((N \vee C) \wedge \neg(N \wedge C)).$

G. We symbolised an *exclusive or* using ‘ \vee ’, ‘ \wedge ’, and ‘ \neg ’. How could you symbolise an *exclusive or* using only two connectives? Is there any way to symbolise an *exclusive or* using only one connective?

For two connectives, we could offer any of the following:

$$\neg(\mathcal{A} \leftrightarrow \mathcal{B})$$

$$(\neg\mathcal{A} \leftrightarrow \mathcal{B})$$

$$(\neg(\neg\mathcal{A} \wedge \neg\mathcal{B}) \wedge \neg(\mathcal{A} \wedge \mathcal{B}))$$

But if we wanted to symbolise it using only one connective, we would have to introduce a new primitive connective.

Sentences of TFL

6

A. For each of the following: (a) Is it a sentence of TFL, strictly speaking?
(b) Is it a sentence of TFL, allowing for our relaxed bracketing conventions?

- | | |
|---|-----------------|
| 1. (A) | (a) no (b) no |
| 2. $J_{374} \vee \neg J_{374}$ | (a) no (b) yes |
| 3. $\neg\neg\neg\neg F$ | (a) yes (b) yes |
| 4. $\neg \wedge S$ | (a) no (b) no |
| 5. $(G \wedge \neg G)$ | (a) yes (b) yes |
| 6. $(A \rightarrow (A \wedge \neg F)) \vee (D \leftrightarrow E)$ | (a) no (b) yes |
| 7. $[(Z \leftrightarrow S) \rightarrow W] \wedge [J \vee X]$ | (a) no (b) yes |
| 8. $(F \leftrightarrow \neg D \rightarrow J) \vee (C \wedge D)$ | (a) no (b) no |

B. Are there any sentences of TFL that contain no atomic sentences? Explain your answer.

No. Atomic sentences contain atomic sentences (trivially). And every more complicated sentence is built up out of less complicated sentences, that were in turn built out of less complicated sentences, . . . , that were ultimately built out of atomic sentences.

C. What is the scope of each connective in the sentence

$$[(H \rightarrow I) \vee (I \rightarrow H)] \wedge (J \vee K)$$

The scope of the left-most instance of ' \rightarrow ' is ' $(H \rightarrow I)$ '.

The scope of the right-most instance of ' \rightarrow ' is ' $(I \rightarrow H)$ '.

The scope of the left-most instance of ' \vee ' is ' $[(H \rightarrow I) \vee (I \rightarrow H)]$ '.

The scope of the right-most instance of ' \vee ' is ' $(J \vee K)$ '.

The scope of the conjunction is the entire sentence; so conjunction is the main logical connective of the sentence.

Complete truth tables

10

A. Complete truth tables for each of the following:

1. $A \rightarrow A$

A	$A \rightarrow A$
T	T
F	T

2. $C \rightarrow \neg C$

C	$C \rightarrow \neg C$
T	F
F	T

3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	$(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$
T	T	F
T	F	T
F	T	T
F	F	F

4. $(A \rightarrow B) \vee (B \rightarrow A)$

A	B	$(A \rightarrow B) \vee (B \rightarrow A)$
T	T	T
T	F	T
F	T	T
F	F	T

5. $(A \wedge B) \rightarrow (B \vee A)$

A	B	$(A \wedge B) \rightarrow (B \vee A)$
T	T	T
T	F	T
F	T	T
F	F	T

6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

A	B	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
T	T	F
T	F	F
F	T	F
F	F	T

7. $[(A \wedge B) \wedge \neg(A \wedge B)] \wedge C$

A	B	C	$[(A \wedge B) \wedge \neg(A \wedge B)] \wedge C$
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

8. $[(A \wedge B) \wedge C] \rightarrow B$

A	B	C	$[(A \wedge B) \wedge C] \rightarrow B$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

9. $\neg[(C \vee A) \vee B]$

A	B	C	$\neg[(C \vee A) \vee B]$
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

B. Check all the claims made in introducing the new notational conventions in §10.3, i.e. show that:

- 1.
- $'((A \wedge B) \wedge C)'$
- and
- $'(A \wedge (B \wedge C))'$
- have the same truth table

A	B	C	$(A \wedge B) \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

- 2.
- $'((A \vee B) \vee C)'$
- and
- $'(A \vee (B \vee C))'$
- have the same truth table

A	B	C	$(A \vee B) \vee C$	$A \vee (B \vee C)$
T	T	T	T T T TT	T T T T T
T	T	F	T T T TF	T T T T F
T	F	T	T T F TT	T T F T T
T	F	F	T T F TF	T T F F F
F	T	T	F T T TT	F T T T T
F	T	F	F T T TF	F T T T F
F	F	T	F F F TT	F T F T T
F	F	F	F F F FF	F F F F F

3. ‘ $((A \vee B) \wedge C)$ ’ and ‘ $(A \vee (B \wedge C))$ ’ do not have the same truth table

A	B	C	$(A \vee B) \wedge C$	$A \vee (B \wedge C)$
T	T	T	T T T TT	T T T T T
T	T	F	T T T FF	T T T F F
T	F	T	T T F TT	T T F F T
T	F	F	T T F FF	T T F F F
F	T	T	F T T TT	F T T T T
F	T	F	F T T FF	F F T F F
F	F	T	F F F FT	F F F F T
F	F	F	F F F FF	F F F F F

4. ‘ $((A \rightarrow B) \rightarrow C)$ ’ and ‘ $(A \rightarrow (B \rightarrow C))$ ’ do not have the same truth table

A	B	C	$(A \rightarrow B) \rightarrow C$	$A \rightarrow (B \rightarrow C)$
T	T	T	T T T TT	T T T T T
T	T	F	T T T FF	T F T F F
T	F	T	T F F TT	T T F T T
T	F	F	T F F TF	T T F T F
F	T	T	F T T TT	F T T T T
F	T	F	F T T FF	F T T F F
F	F	T	F T F TT	F T F T T
F	F	F	F T F FF	F T F T F

Also, check whether:

5. ‘ $((A \leftrightarrow B) \leftrightarrow C)$ ’ and ‘ $(A \leftrightarrow (B \leftrightarrow C))$ ’ have the same truth table
Indeed they do:

A	B	C	$(A \leftrightarrow B) \leftrightarrow C$	$A \leftrightarrow (B \leftrightarrow C)$
T	T	T	T T T TT	T T T T T
T	T	F	T T T FF	T F T F F
T	F	T	T F F FT	T F F F T
T	F	F	T F F TF	T T F T F
F	T	T	F F T FT	F F T T T
F	T	F	F F T TF	F T T F F
F	F	T	F T F TT	F T F F T
F	F	F	F T F FF	F F F T F

Semantic concepts

11

A. Revisit your answers to §10A. Determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

- | | |
|---|---------------|
| 1. $A \rightarrow A$ | Tautology |
| 2. $C \rightarrow \neg C$ | Neither |
| 3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$ | Tautology |
| 4. $(A \rightarrow B) \vee (B \rightarrow A)$ | Tautology |
| 5. $(A \wedge B) \rightarrow (B \vee A)$ | Tautology |
| 6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ | Tautology |
| 7. $[(A \wedge B) \wedge \neg(A \wedge B)] \wedge C$ | Contradiction |
| 8. $[(A \wedge B) \wedge C] \rightarrow B$ | Tautology |
| 9. $\neg[(C \vee A) \vee B]$ | Neither |

B. Use truth tables to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \rightarrow A, \neg A \rightarrow \neg A, A \wedge A, A \vee A$ Jointly consistent (see line 1)

A	$A \rightarrow A$	$\neg A \rightarrow \neg A$	$A \wedge A$	$A \vee A$
T	T TT	FT TFF	T TT	T TT
F	F TFF	TF TTF	F FF	F FF

2. $A \vee B, A \rightarrow C, B \rightarrow C$ Jointly consistent (see line 1)

A	B	C	$A \vee B$	$A \rightarrow C$	$B \rightarrow C$
T	T	T	T TT	T TT	T TT
T	T	F	T TT	T FF	T FF
T	F	T	T TT	T TT	F TT
T	F	F	T TFF	T FF	F TFF
F	T	T	F TFF	F TT	T TT
F	T	F	F TTF	F TFF	T TFF
F	F	T	F FFF	F TT	F TT
F	F	F	F FFF	F TFF	F TFF

3. $B \wedge (C \vee A), A \rightarrow B, \neg(B \vee C)$ Jointly inconsistent

A	B	C	$B \wedge (C \vee A)$	$A \rightarrow B$	$\neg (B \vee C)$
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	F	F	F
T	F	F	F	F	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	F	F
F	F	F	F	F	T

4. $A \leftrightarrow (B \vee C), C \rightarrow \neg A, A \rightarrow \neg B$ Jointly consistent (see line 8)

A	B	C	$A \leftrightarrow (B \vee C)$	$C \rightarrow \neg A$	$A \rightarrow \neg B$
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	T	F

C. Use truth tables to determine whether each argument is valid or invalid.

1. $A \rightarrow A \therefore A$ Invalid (see line 2)

A	$A \rightarrow A$	A
T	T	T
F	F	F

2. $A \rightarrow (A \wedge \neg A) \therefore \neg A$ Valid

A	$A \rightarrow (A \wedge \neg A)$	$\neg A$
T	F	F
F	T	T

3. $A \vee (B \rightarrow A) \therefore \neg A \rightarrow \neg B$ Valid

A	B	$A \vee (B \rightarrow A)$	$\neg A \rightarrow \neg B$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	F	T

4. $A \vee B, B \vee C, \neg A \therefore B \wedge C$ Invalid (see line 6)

A	B	C	$A \vee B$	$B \vee C$	$\neg A$	$B \wedge C$
T	T	T	T	T	F	T
T	T	F	T	T	F	T
T	F	T	T	F	F	F
T	F	F	T	F	F	F
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	F	F	F	F
T	F	F	F	F	F	F

5. $(B \wedge A) \rightarrow C, (C \wedge A) \rightarrow B \therefore (C \wedge B) \rightarrow A$ Invalid (see line 5)

A	B	C	$(B \wedge A) \rightarrow C$	$(C \wedge A) \rightarrow B$	$(C \wedge B) \rightarrow A$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	F
F	T	F	T	F	F
F	F	T	F	F	F
F	F	F	F	F	F

D. Answer each of the questions below and justify your answer.

- Suppose that \mathcal{A} and \mathcal{B} are tautologically equivalent. What can you say about $\mathcal{A} \leftrightarrow \mathcal{B}$?
 \mathcal{A} and \mathcal{B} have the same truth value on every line of a complete truth table, so $\mathcal{A} \leftrightarrow \mathcal{B}$ is true on every line. It is a tautology.
- Suppose that $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$ is neither a tautology nor a contradiction. What can you say about whether $\mathcal{A}, \mathcal{B} \therefore \mathcal{C}$ is valid?
 Since the sentence $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, \mathcal{A} and \mathcal{B} are true and \mathcal{C} is false. So the argument is invalid.
- Suppose that \mathcal{A}, \mathcal{B} and \mathcal{C} are jointly tautologically inconsistent. What can you say about $(\mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C})$?
 Since the sentences are jointly tautologically inconsistent, there is no valuation on which they are all true. So their conjunction is false on every valuation. It is a contradiction.
- Suppose that \mathcal{A} is a contradiction. What can you say about whether $\mathcal{A}, \mathcal{B} \vDash \mathcal{C}$?
 Since \mathcal{A} is false on every line of a complete truth table, there is no line on which \mathcal{A} and \mathcal{B} are true and \mathcal{C} is false. So the entailment holds.
- Suppose that \mathcal{C} is a tautology. What can you say about whether $\mathcal{A}, \mathcal{B} \vDash \mathcal{C}$?
 Since \mathcal{C} is true on every line of a complete truth table, there is no line on which \mathcal{A} and \mathcal{B} are true and \mathcal{C} is false. So the entailment holds.
- Suppose that \mathcal{A} and \mathcal{B} are tautologically equivalent. What can you say about $(\mathcal{A} \vee \mathcal{B})$?
 Not much. Since \mathcal{A} and \mathcal{B} are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is tautologically equivalent to them.
- Suppose that \mathcal{A} and \mathcal{B} are *not* tautologically equivalent. What can you say about $(\mathcal{A} \vee \mathcal{B})$?
 \mathcal{A} and \mathcal{B} have different truth values on at least one line of a complete truth table, and $(\mathcal{A} \vee \mathcal{B})$ will be true on that line. On other lines, it might be true or false. So $(\mathcal{A} \vee \mathcal{B})$ is either a tautology or it is contingent; it is *not* a contradiction.

E. Consider the following principle:

- Suppose \mathcal{A} and \mathcal{B} are tautologically equivalent. Suppose an argument contains \mathcal{A} (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced \mathcal{A} with \mathcal{B} .

Is this principle correct? Explain your answer.

The principle is correct. Since \mathcal{A} and \mathcal{B} are tautologically equivalent, they have the same truth table. So every valuation that makes \mathcal{A} true also makes \mathcal{B} true, and every valuation that makes \mathcal{A} false also makes \mathcal{B} false. So if no valuation makes all the premises true and the conclusion false, when \mathcal{A} was among the premises or the conclusion, then no valuation makes all the premises true and the conclusion false, when we replace \mathcal{A} with \mathcal{B} .

Truth table shortcuts

12

A. Using shortcuts, determine whether each sentence is a tautology, a contradiction, or neither.

1. $\neg B \wedge B$ Contradiction

B	$\neg B \wedge B$
T	F F
F	F

2. $\neg D \vee D$ Tautology

D	$\neg D \vee D$
T	T
F	T T

3. $(A \wedge B) \vee (B \wedge A)$ Neither

A	B	$(A \wedge B) \vee (B \wedge A)$
T	T	T T
T	F	F F F
F	T	F F F
F	F	F F F

4. $\neg[A \rightarrow (B \rightarrow A)]$ Contradiction

A	B	$\neg[A \rightarrow (B \rightarrow A)]$
T	T	F T T
T	F	F T T
F	T	F T
F	F	F T

5. $A \leftrightarrow [A \rightarrow (B \wedge \neg B)]$ Contradiction

A	B	$A \leftrightarrow [A \rightarrow (B \wedge \neg B)]$
T	T	F F FF
T	F	F F F
F	T	F T
F	F	F T

6. $\neg(A \wedge B) \leftrightarrow A$ Neither

A	B	$\neg(A \wedge B) \leftrightarrow A$		
T	T	F	T	F
T	F	T	F	T
F	T	T	F	F
F	F	T	F	F

7. $A \rightarrow (B \vee C)$

Neither

A	B	C	$A \rightarrow (B \vee C)$	
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	
F	T	F	T	
F	F	T	T	
F	F	F	T	

8. $(A \wedge \neg A) \rightarrow (B \vee C)$

Tautology

A	B	C	$(A \wedge \neg A) \rightarrow (B \vee C)$	
T	T	T	FF	T
T	T	F	FF	T
T	F	T	FF	T
T	F	F	FF	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

9. $(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$

Neither

A	B	C	D	$(B \wedge D) \leftrightarrow [A \leftrightarrow (A \vee C)]$			
T	T	T	T	T	T	T	T
T	T	T	F	F	F	T	T
T	T	F	T	T	T	T	T
T	T	F	F	F	F	T	T
T	F	T	T	F	F	T	T
T	F	T	F	F	F	T	T
T	F	F	T	F	F	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	F	F	T
F	T	T	F	F	T	F	T
F	T	F	T	T	T	T	F
F	T	F	F	F	F	T	F
F	F	T	T	F	T	F	T
F	F	T	F	F	T	F	T
F	F	F	T	F	F	T	F
F	F	F	F	F	F	T	F

Partial truth tables

A. Use complete or partial truth tables (as appropriate) to determine whether these pairs of sentences are tautologically equivalent:

1. $A, \neg A$ Not tautologically equivalent

A	A	$\neg A$
T	T	F

2. $A, A \vee A$ Tautologically equivalent

A	A	$A \vee A$
T	T	T
T	T	T

3. $A \rightarrow A, A \leftrightarrow A$ Tautologically equivalent

A	$A \rightarrow A$	$A \leftrightarrow A$
T	T	T
F	T	T

4. $A \vee \neg B, A \rightarrow B$ Not tautologically equivalent

A	B	$A \vee \neg B$	$A \rightarrow B$
T	F	T	F

5. $A \wedge \neg A, \neg B \leftrightarrow B$ Tautologically equivalent

A	B	$A \wedge \neg A$	$\neg B \leftrightarrow B$
T	T	F	F
T	F	F	T
F	T	F	F
F	F	F	T

6. $\neg(A \wedge B), \neg A \vee \neg B$ Tautologically equivalent

A	B	$\neg(A \wedge B)$	$\neg A \vee \neg B$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

7. $\neg(A \rightarrow B), \neg A \rightarrow \neg B$ Not tautologically equivalent

A	B	$\neg(A \rightarrow B)$	$\neg A \rightarrow \neg B$
T	T	F	F

- 8.
- $(A \rightarrow B), (\neg B \rightarrow \neg A)$
- Tautologically equivalent

A	B	$(A \rightarrow B)$	$(\neg B \rightarrow \neg A)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	T

B. Use complete or partial truth tables (as appropriate) to determine whether these sentences are jointly tautologically consistent, or jointly tautologically inconsistent:

- 1.
- $A \wedge B, C \rightarrow \neg B, C$
- Jointly tautologically inconsistent

A	B	C	$A \wedge B$	$C \rightarrow \neg B$	C
T	T	T	T	F	T
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	F	F	T
F	T	F	F	T	F
F	F	T	F	T	T
F	F	F	F	T	F

- 2.
- $A \rightarrow B, B \rightarrow C, A, \neg C$
- Jointly tautologically inconsistent

A	B	C	$A \rightarrow B$	$B \rightarrow C$	A	$\neg C$
T	T	T	T	T	T	F
T	T	F	T	F	T	T
T	F	T	F	T	T	F
T	F	F	F	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	F	T
F	F	T	T	T	F	F
F	F	F	T	T	F	T

- 3.
- $A \vee B, B \vee C, C \rightarrow \neg A$
- Jointly tautologically consistent

A	B	C	$A \vee B$	$B \vee C$	$C \rightarrow \neg A$
F	T	T	T	T	T

- 4.
- $A, B, C, \neg D, \neg E, F$
- Jointly tautologically consistent

A	B	C	D	E	F	A	B	C	$\neg D$	$\neg E$	F
T	T	T	F	F	T	T	T	T	T	T	T

C. Use complete or partial truth tables (as appropriate) to determine whether each argument is valid or invalid:

- 1.
- $A \vee [A \rightarrow (A \leftrightarrow A)] \therefore A$
- Invalid

A	$A \vee [A \rightarrow (A \leftrightarrow A)]$	A
F	T	F

2. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$

Invalid

A	B	$A \leftrightarrow \neg(B \leftrightarrow A)$	A
F	F	TF	T

3. $A \rightarrow B, B \therefore A$

Invalid

A	B	$A \rightarrow B$	B	A
F	T	T	T	F

4. $A \vee B, B \vee C, \neg B \therefore A \wedge C$

Valid

A	B	C	$A \vee B$	$B \vee C$	$\neg B$	$A \wedge C$
T	T	T				T
T	T	F			F	F
T	F	T				T
T	F	F	T	F	T	F
F	T	T			F	F
F	T	F			F	F
F	F	T	F		T	F
F	F	F	F		T	F

5. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$

Valid

A	B	C	$A \leftrightarrow B$	$B \leftrightarrow C$	$A \leftrightarrow C$
T	T	T			T
T	T	F	T	F	F
T	F	T			T
T	F	F	F		F
F	T	T	F		F
F	T	F			T
F	F	T	T	F	F
F	F	F			T

Sentences with one quantifier

15

A. Here are the syllogistic figures identified by Aristotle and his successors, along with their medieval names:

- **Barbara.** All G are F. All H are G. So: All H are F
 $\forall x(Gx \rightarrow Fx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow Fx)$
- **Celarent.** No G are F. All H are G. So: No H are F
 $\forall x(Gx \rightarrow \neg Fx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Ferio.** No G are F. Some H is G. So: Some H is not F
 $\forall x(Gx \rightarrow \neg Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Darii.** All G are H. Some H is G. So: Some H is F.
 $\forall x(Gx \rightarrow Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge Fx)$
- **Camestres.** All F are G. No H are G. So: No H are F.
 $\forall x(Fx \rightarrow Gx), \forall x(Hx \rightarrow \neg Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Cesare.** No F are G. All H are G. So: No H are F.
 $\forall x(Fx \rightarrow \neg Gx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Baroko.** All F are G. Some H is not G. So: Some H is not F.
 $\forall x(Fx \rightarrow Gx), \exists x(Hx \wedge \neg Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Festino.** No F are G. Some H are G. So: Some H is not F.
 $\forall x(Fx \rightarrow \neg Gx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Datisi.** All G are F. Some G is H. So: Some H is F.
 $\forall x(Gx \rightarrow Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge Fx)$
- **Disamis.** Some G is F. All G are H. So: Some H is F.
 $\exists x(Gx \wedge Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$
- **Ferison.** No G are F. Some G is H. So: Some H is not F.
 $\forall x(Gx \rightarrow \neg Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Bokardo.** Some G is not F. All G are H. So: Some H is not F.
 $\exists x(Gx \wedge \neg Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Camenes.** All F are G. No G are H So: No H is F.
 $\forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow \neg Hx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Dimaris.** Some F is G. All G are H. So: Some H is F.
 $\exists x(Fx \wedge Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$
- **Fresison.** No F are G. Some G is H. So: Some H is not F.
 $\forall x(Fx \rightarrow \neg Gx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge \neg Fx)$

Symbolise each argument in FOL.

B. Using the following symbolisation key:

domain: people

Kx : ______x knows the combination to the safe

Sx : _____ x is a spy
 Vx : _____ x is a vegetarian
 h : Hofthor
 i : Ingmar

symbolise the following sentences in FOL:

- Neither Hofthor nor Ingmar is a vegetarian.
 $\neg Vh \wedge \neg Vi$
- No spy knows the combination to the safe.
 $\forall x(Sx \rightarrow \neg Kx)$
- No one knows the combination to the safe unless Ingmar does.
 $\forall x\neg Kx \vee Ki$
- Hofthor is a spy, but no vegetarian is a spy.
 $Sh \wedge \forall x(Vx \rightarrow \neg Sx)$

C. Using this symbolisation key:

domain: all animals

Ax : _____ x is an alligator.
 Mx : _____ x is a monkey.
 Rx : _____ x is a reptile.
 Zx : _____ x lives at the zoo.
 a : Amos
 b : Bouncer
 c : Cleo

symbolise each of the following sentences in FOL:

- Amos, Bouncer, and Cleo all live at the zoo.
 $Za \wedge Zb \wedge Zc$
- Bouncer is a reptile, but not an alligator.
 $Rb \wedge \neg Ab$
- Some reptile lives at the zoo.
 $\exists x(Rx \wedge Zx)$
- Every alligator is a reptile.
 $\forall x(Ax \rightarrow Rx)$
- Any animal that lives at the zoo is either a monkey or an alligator.
 $\forall x(Zx \rightarrow (Mx \vee Ax))$
- There are reptiles which are not alligators.
 $\exists x(Rx \wedge \neg Ax)$
- If any animal is a reptile, then Amos is.
 $\exists xRx \rightarrow Ra$
- If any animal is an alligator, then it is a reptile.
 $\forall x(Ax \rightarrow Rx)$

D. For each argument, write a symbolisation key and symbolise the argument in FOL.

- Willard is a logician. All logicians wear funny hats. So Willard wears a funny hat
domain: people

Lx : _____ x is a logician
 Hx : _____ x wears a funny hat
 i : Willard

$Li, \forall x(Lx \rightarrow Hx) \therefore Hi$

2. Nothing on my desk escapes my attention. There is a computer on my desk. As such, there is a computer that does not escape my attention.

domain: physical things

Dx : _____ x is on my desk
 Ex : _____ x escapes my attention
 Cx : _____ x is a computer

$\forall x(Dx \rightarrow \neg Ex), \exists x(Dx \wedge Cx) \therefore \exists x(Cx \wedge \neg Ex)$

3. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.

domain: episodes (psychological and televised)

Dx : _____ x is one of my dreams
 Bx : _____ x is in black and white
 Ox : _____ x is an old TV show

$\forall x(Dx \rightarrow Bx), \forall x(Ox \rightarrow Bx) \therefore \exists x(Dx \wedge Ox)$.

Comment: generic statements are tricky to deal with. Does the second sentence mean that *all* old TV shows are in black and white; or that most of them are; or that most of the things which are in black and white are old TV shows? I have gone with the former, but it is not clear that FOL deals with these well.

4. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.

domain: people

Ax : _____ x has been to Australia
 Kx : _____ x has seen a kangaroo
 Zx : _____ x has been to a zoo
 h : Holmes
 a : Watson

$\neg Ah \wedge \neg Aa, \forall x(Kx \rightarrow (Ax \vee Zx)), \neg Ka \wedge Kh \therefore Zh$

5. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.

domain: people

Sx : _____ x expects the Spanish Inquisition
 Tx : _____ x knows the troubles I've seen
 h : Holmes
 a : Watson

$\forall x \neg Sx, \forall x \neg Tx \therefore \forall x(Sx \rightarrow Tx)$

6. All babies are illogical. Nobody who is illogical can manage a crocodile. Berthold is a baby. Therefore, Berthold is unable to manage a crocodile.

domain: people

Bx : _____ x is a baby

Ix : _____ x is illogical

Cx : _____ x can manage a crocodile

b : Berthold

$\forall x(Bx \rightarrow Ix), \forall x(Ix \rightarrow \neg Cx), Bb \therefore \neg Cb$

Multiple generality

16

A. Using this symbolisation key:

domain: all animals

Ax : _____ x is an alligator

Mx : _____ x is a monkey

Rx : _____ x is a reptile

Zx : _____ x lives at the zoo

Lxy : _____ x loves _____ y

a : Amos

b : Bouncer

c : Cleo

symbolise each of the following sentences in FOL:

1. If Cleo loves Bouncer, then Bouncer is a monkey.

$Lcb \rightarrow Mb$

2. If both Bouncer and Cleo are alligators, then Amos loves them both.

$(Ab \wedge Ac) \rightarrow (Lab \wedge Lac)$

3. Cleo loves a reptile.

$\exists x(Rx \wedge Lcx)$

Comment: this English expression is ambiguous; in some contexts, it can be read as a generic, along the lines of 'Cleo loves reptiles'. (Compare 'I do love a good pint'.)

4. Bouncer loves all the monkeys that live at the zoo.

$\forall x((Mx \wedge Zx) \rightarrow Lbx)$

5. All the monkeys that Amos loves love him back.

$\forall x((Mx \wedge Lax) \rightarrow Lxa)$

6. Every monkey that Cleo loves is also loved by Amos.

$\forall x((Mx \wedge Lcx) \rightarrow Lax)$

7. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.

$\exists x(Mx \wedge Lxb \wedge \neg Lbx)$

B. Using the following symbolisation key:

domain: all animals

Dx : _____ x is a dog

Sx : _____ x likes samurai movies

Lxy : _____ x is larger than _____ y

b : Bertie

e : Emerson

f : Fergis

symbolise the following sentences in FOL:

1. Bertie is a dog who likes samurai movies.
 $Db \wedge Sb$
2. Bertie, Emerson, and Fergis are all dogs.
 $Db \wedge De \wedge Df$
3. Emerson is larger than Bertie, and Fergis is larger than Emerson.
 $Leb \wedge Lfe$
4. All dogs like samurai movies.
 $\forall x(Dx \rightarrow Sx)$
5. Only dogs like samurai movies.
 $\forall x(Sx \rightarrow Dx)$
Comment: the FOL sentence just written does not require that anyone likes samurai movies. The English sentence might suggest that at least some dogs *do* like samurai movies?
6. There is a dog that is larger than Emerson.
 $\exists x(Dx \wedge Lxe)$
7. If there is a dog larger than Fergis, then there is a dog larger than Emerson.
 $\exists x(Dx \wedge Lxf) \rightarrow \exists x(Dx \wedge Lxe)$
8. No animal that likes samurai movies is larger than Emerson.
 $\forall x(Sx \rightarrow \neg Lxe)$
9. No dog is larger than Fergis.
 $\forall x(Dx \rightarrow \neg Lxf)$
10. Any animal that dislikes samurai movies is larger than Bertie.
 $\forall x(\neg Sx \rightarrow Lxb)$
Comment: this is very poor, though! For ‘dislikes’ does not mean the same as ‘does not like’.
11. There is an animal that is between Bertie and Emerson in size.
 $\exists x((Lbx \wedge Lxe) \vee (Lex \wedge Lxb))$
12. There is no dog that is between Bertie and Emerson in size.
 $\forall x(Dx \rightarrow \neg[(Lbx \wedge Lxe) \vee (Lex \wedge Lxb)])$
13. No dog is larger than itself.
 $\forall x(Dx \rightarrow \neg Lxx)$
14. Every dog is larger than some dog.
 $\forall x(Dx \rightarrow \exists y(Dy \wedge Lxy))$
Comment: the English sentence is potentially ambiguous here. I have resolved the ambiguity by assuming it should be paraphrased by ‘for every dog, there is a dog smaller than it’.
15. There is an animal that is smaller than every dog.
 $\exists x\forall y(Dy \rightarrow Lyx)$
16. If there is an animal that is larger than any dog, then that animal does not like samurai movies.
 $\forall x(\forall y(Dy \rightarrow Lxy) \rightarrow \neg Sx)$
Comment: I have assumed that ‘larger than any dog’ here means ‘larger than every dog’.

C. Using the following symbolisation key:

domain: people and dishes at a potluck

Rx : _____ x has run out.

Tx : _____ x is on the table.

Fx : _____ x is food.

Px : _____ x is a person.

Lxy : _____ x likes _____ y .

e : Eli

f : Francesca

g : the guacamole

symbolise the following English sentences in FOL:

1. All the food is on the table.
 $\forall x(Fx \rightarrow Tx)$
2. If the guacamole has not run out, then it is on the table.
 $\neg Rg \rightarrow Tg$
3. Everyone likes the guacamole.
 $\forall x(Px \rightarrow Lxg)$
4. If anyone likes the guacamole, then Eli does.
 $\exists x(Px \wedge Lxg) \rightarrow Leg$
5. Francesca only likes the dishes that have run out.
 $\forall x[(Lfx \wedge Fx) \rightarrow Rx]$
6. Francesca likes no one, and no one likes Francesca.
 $\forall x[Px \rightarrow (\neg Lfx \wedge \neg Lxf)]$
7. Eli likes anyone who likes the guacamole.
 $\forall x((Px \wedge Lxg) \rightarrow Lex)$
8. Eli likes anyone who likes the people that he likes.
 $\forall x[(Px \wedge \forall y[(Py \wedge Ley) \rightarrow Lxy]) \rightarrow Lex]$
9. If there is a person on the table already, then all of the food must have run out.
 $\exists x(Px \wedge Tx) \rightarrow \forall x(Fx \rightarrow Rx)$

D. Using the following symbolisation key:

domain: people

Dx : _____ x dances ballet.

Fx : _____ x is female.

Mx : _____ x is male.

Cxy : _____ x is a child of _____ y .

Sxy : _____ x is a sibling of _____ y .

e : Elmer

j : Jane

p : Patrick

symbolise the following arguments in FOL:

1. All of Patrick's children are ballet dancers.
 $\forall x(Cxp \rightarrow Dx)$
2. Jane is Patrick's daughter.
 $Cjp \wedge Fj$
3. Patrick has a daughter.
 $\exists x(Cxp \wedge Fx)$

4. Jane is an only child.
 $\neg\exists xSxj$
5. All of Patrick's sons dance ballet.
 $\forall x[(Cxp \wedge Mx) \rightarrow Dx]$
6. Patrick has no sons.
 $\neg\exists x(Cxp \wedge Mx)$
7. Jane is Elmer's niece.
 $\exists x(Sxe \wedge Cjx \wedge Fj)$
8. Patrick is Elmer's brother.
 $Spe \wedge Mp$
9. Patrick's brothers have no children.
 $\forall x[(Sp x \wedge Mx) \rightarrow \neg\exists yCyx]$
10. Jane is an aunt.
 $Fj \wedge \exists x(Sxj \wedge \exists yCyx)$
11. Everyone who dances ballet has a brother who also dances ballet.
 $\forall x[Dx \rightarrow \exists y(My \wedge Syx \wedge Dy)]$
12. Every woman who dances ballet is the child of someone who dances ballet.
 $\forall x[(Fx \wedge Dx) \rightarrow \exists y(Cxy \wedge Dy)]$

Identity

17

A. Explain why:

- ‘ $\exists x \forall y (Ay \leftrightarrow x = y)$ ’ is a good symbolisation of ‘there is exactly one apple’.

We might naturally read this in English thus:

- There is something, x , such that, if you choose any object at all, if you chose an apple then you chose x itself, and if you chose x itself then you chose an apple.

The x in question must therefore be the one and only thing which is an apple.

- ‘ $\exists x \exists y [\neg x = y \wedge \forall z (Az \leftrightarrow (x = z \vee y = z))]$ ’ is a good symbolisation of ‘there are exactly two apples’.

Similarly to the above, we might naturally read this in English thus:

- There are two distinct things, x and y , such that if you choose any object at all, if you chose an apple then you either chose x or y , and if you chose either x or y then you chose an apple.

The x and y in question must therefore be the only things which are apples, and since they are distinct, there are two of them.

Definite descriptions

18

A. Using the following symbolisation key:

domain: people

Kx : _____ x knows the combination to the safe.

Sx : _____ x is a spy.

Vx : _____ x is a vegetarian.

Txy : _____ x trusts _____ y .

h : Hofthor

i : Ingmar

symbolise the following sentences in FOL:

1. Hofthor trusts a vegetarian.

$$\exists x(Vx \wedge Thx)$$

2. Everyone who trusts Ingmar trusts a vegetarian.

$$\forall x[Txi \rightarrow \exists y(Txy \wedge Vy)]$$

3. Everyone who trusts Ingmar trusts someone who trusts a vegetarian.

$$\forall x[Txi \rightarrow \exists y(Txy \wedge \exists z(Tyz \wedge Vz))]$$

4. Only Ingmar knows the combination to the safe.

$$\forall x(Ki \rightarrow x = i)$$

Comment: does the English claim entail that Ingmar *does* know the combination to the safe? If so, then we should formalise this with a ' \leftrightarrow '.

5. Ingmar trusts Hofthor, but no one else.

$$\forall x(Tix \leftrightarrow x = h)$$

6. The person who knows the combination to the safe is a vegetarian.

$$\exists x[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge Vx]$$

7. The person who knows the combination to the safe is not a spy.

$$\exists x[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge \neg Sx]$$

Comment: the scope of negation is potentially ambiguous here; I have read it as *inner* negation.

B. Using the following symbolisation key:

domain: cards in a standard deck

Bx : _____ x is black.

Cx : _____ x is a club.

Dx : _____ x is a deuce.

Jx : _____ x is a jack.

Mx : _____ x is a man with an axe.

Ox : _____ x is one-eyed.

Wx : _____ x is wild.

symbolise each sentence in FOL:

1. All clubs are black cards.

$$\forall x(Cx \rightarrow Bx)$$

2. There are no wild cards.

$$\neg \exists x Wx$$

3. There are at least two clubs.

$$\exists x \exists y (\neg x = y \wedge Cx \wedge Cy)$$

4. There is more than one one-eyed jack.

$$\exists x \exists y (\neg x = y \wedge Jx \wedge Ox \wedge Jy \wedge Oy)$$

5. There are at most two one-eyed jacks.

$$\forall x \forall y \forall z [(Jx \wedge Ox \wedge Jy \wedge Oy \wedge Jz \wedge Oz) \rightarrow (x = y \vee x = z \vee y = z)]$$

6. There are two black jacks.

$$\exists x \exists y (\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy)$$

Comment: I am reading this as ‘there are *at least* two...’. If the suggestion was that there are *exactly* two, then a different FOL sentence would be required, namely:

$$\exists x \exists y (\neg x = y \wedge Bx \wedge Jx \wedge By \wedge Jy \wedge \forall z [(Bz \wedge Jz) \rightarrow (x = z \vee y = z)])$$

7. There are four deuces.

$$\exists w \exists x \exists y \exists z (\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz)$$

Comment: I am reading this as ‘there are *at least* four...’. If the suggestion is that there are *exactly* four, then we should offer instead:

$$\exists w \exists x \exists y \exists z (\neg w = x \wedge \neg w = y \wedge \neg w = z \wedge \neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Dw \wedge Dx \wedge Dy \wedge Dz \wedge \forall v [Dv \rightarrow (v = w \vee v = x \vee v = y \vee v = z)])$$

8. The deuce of clubs is a black card.

$$\exists x [Dx \wedge Cx \wedge \forall y ((Dy \wedge Cy) \rightarrow x = y) \wedge Bx]$$

9. One-eyed jacks and the man with the axe are wild.

$$\forall x [(Jx \wedge Ox) \rightarrow Wx] \wedge \exists x [Mx \wedge \forall y (My \rightarrow x = y) \wedge Wx]$$

10. If the deuce of clubs is wild, then there is exactly one wild card.

$$\exists x (Dx \wedge Cx \wedge \forall y [(Dy \wedge Cy) \rightarrow x = y] \wedge Wx) \rightarrow \exists x (Wx \wedge \forall y (Wy \rightarrow x = y))$$

Comment: if there is not exactly one deuce of clubs, then the above sentence is true. Maybe that’s the wrong verdict. Perhaps the sentence should definitely be taken to imply that there is one and only one deuce of clubs, and then express a conditional about wildness. If so, then we might symbolise it thus:

$$\exists x (Dx \wedge Cx \wedge \forall y [(Dy \wedge Cy) \rightarrow x = y] \wedge [Wx \rightarrow \forall y (Wy \rightarrow x = y)])$$

11. The man with the axe is not a jack.

$$\exists x [Mx \wedge \forall y (My \rightarrow x = y) \wedge \neg Jx]$$

12. The deuce of clubs is not the man with the axe.

$$\exists x \exists y (Dx \wedge Cx \wedge \forall z [(Dz \wedge Cz) \rightarrow x = z] \wedge My \wedge \forall z (Mz \rightarrow y = x) \wedge \neg x = y)$$

C. Using the following symbolisation key:

domain: animals in the world

Bx : _____ x is in Farmer Brown’s field.

Hx : _____ x is a horse.

Px : _____ x is a Pegasus.

Wx : _____ x has wings.

symbolise the following sentences in FOL:

1. There are at least three horses in the world.
 $\exists x \exists y \exists z (\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Hx \wedge Hy \wedge Hz)$
2. There are at least three animals in the world.
 $\exists x \exists y \exists z (\neg x = y \wedge \neg x = z \wedge \neg y = z)$
3. There is more than one horse in Farmer Brown's field.
 $\exists x \exists y (\neg x = y \wedge Hx \wedge Hy \wedge Bx \wedge By)$
4. There are three horses in Farmer Brown's field.
 $\exists x \exists y \exists z (\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge Hx \wedge Hy \wedge Hz \wedge Bx \wedge By \wedge Bz)$
 Comment: I have read this as 'there are *at least* three...'. If the suggestion was that there are *exactly* three, then a different FOL sentence would be required.
5. There is a single winged creature in Farmer Brown's field; any other creatures in the field must be wingless.
 $\exists x [Wx \wedge Bx \wedge \forall y ((Wy \wedge By) \rightarrow x = y)]$
6. The Pegasus is a winged horse.
 $\exists x [Px \wedge \forall y (Py \rightarrow x = y) \wedge Wx \wedge Hx]$
7. The animal in Farmer Brown's field is not a horse.
 $\exists x [Bx \wedge \forall y (By \rightarrow x = y) \wedge \neg Hx]$
 Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.
8. The horse in Farmer Brown's field does not have wings.
 $\exists x [Hx \wedge Bx \wedge \forall y ((Hy \wedge By) \rightarrow x = y) \wedge \neg Wx]$
 Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.

D. In this section, I symbolised 'Nick is the traitor' by ' $\exists x (Tx \wedge \forall y (Ty \rightarrow x = y) \wedge x = n)$ '. Two equally good symbolisations would be:

- $Tn \wedge \forall y (Ty \rightarrow n = y)$
 This sentence requires that Nick is a traitor, and that Nick alone is a traitor. Otherwise put, there is one and only one traitor, namely, Nick. Otherwise put: Nick is the traitor.
- $\forall y (Ty \leftrightarrow y = n)$
 This sentence can be understood thus: Take anything you like; now, if you chose a traitor, you chose Nick, and if you chose Nick, you chose a traitor. So there is one and only one traitor, namely, Nick, as required.

Explain why these would be equally good symbolisations.

Sentences of FOL

19

A. Identify which variables are bound and which are free. I shall underline the bound variables, and put free variables in blue.

1. $\exists x L\underline{xy} \wedge \forall y L\underline{yx}$
2. $\forall x A\underline{x} \wedge Bx$
3. $\forall x (A\underline{x} \wedge B\underline{x}) \wedge \forall y (Cx \wedge D\underline{y})$
4. $\forall x \exists y [R\underline{xy} \rightarrow (Jz \wedge K\underline{x})] \vee R\underline{yx}$
5. $\forall x_1 (M\underline{x_2} \leftrightarrow L\underline{x_2x_1}) \wedge \exists x_2 L\underline{x_3x_2}$

Truth in FOL

A. Consider the following interpretation:

- The domain comprises only Corwin and Benedict
- ‘ Ax ’ is to be true of both Corwin and Benedict
- ‘ Bx ’ is to be true of Benedict only
- ‘ Nx ’ is to be true of no one
- ‘ c ’ is to refer to Corwin

Determine whether each of the following sentences is true or false in that interpretation:

- | | |
|--|-------|
| 1. Bc | False |
| 2. $Ac \leftrightarrow \neg Nc$ | True |
| 3. $Nc \rightarrow (Ac \vee Bc)$ | True |
| 4. $\forall xAx$ | True |
| 5. $\forall x\neg Bx$ | False |
| 6. $\exists x(Ax \wedge Bx)$ | True |
| 7. $\exists x(Ax \rightarrow Nx)$ | False |
| 8. $\forall x(Nx \vee \neg Nx)$ | True |
| 9. $\exists xBx \rightarrow \forall xAx$ | True |

B. Consider the following interpretation:

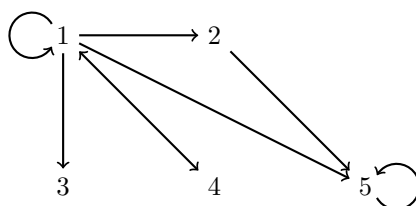
- The domain comprises only Lemmy, Courtney and Eddy
- ‘ Gx ’ is to be true of Lemmy, Courtney and Eddy.
- ‘ Hx ’ is to be true of and only of Courtney
- ‘ Mx ’ is to be true of and only of Lemmy and Eddy
- ‘ c ’ is to refer to Courtney
- ‘ e ’ is to refer to Eddy

Determine whether each of the following sentences is true or false in that interpretation:

- | | |
|------------------------------|-------|
| 1. Hc | True |
| 2. He | False |
| 3. $Mc \vee Me$ | True |
| 4. $Gc \vee \neg Gc$ | True |
| 5. $Mc \rightarrow Gc$ | True |
| 6. $\exists xHx$ | True |
| 7. $\forall xHx$ | False |
| 8. $\exists x\neg Mx$ | True |
| 9. $\exists x(Hx \wedge Gx)$ | True |

- | | |
|---|-------|
| 10. $\exists x(Mx \wedge Gx)$ | True |
| 11. $\forall x(Hx \vee Mx)$ | True |
| 12. $\exists xHx \wedge \exists xMx$ | True |
| 13. $\forall x(Hx \leftrightarrow \neg Mx)$ | True |
| 14. $\exists xGx \wedge \exists x\neg Gx$ | False |
| 15. $\forall x\exists y(Gx \wedge Hy)$ | True |

C. Following the diagram conventions introduced at the end of §23, consider the following interpretation:



Determine whether each of the following sentences is true or false in that interpretation:

- | | |
|---|-------|
| 1. $\exists xRxx$ | True |
| 2. $\forall xRxx$ | False |
| 3. $\exists x\forall yRxy$ | True |
| 4. $\exists x\forall yRyx$ | False |
| 5. $\forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz)$ | False |
| 6. $\forall x\forall y\forall z((Rxy \wedge Rxz) \rightarrow Ryz)$ | False |
| 7. $\exists x\forall y\neg Rxy$ | True |
| 8. $\forall x(\exists yRxy \rightarrow \exists yRyx)$ | True |
| 9. $\exists x\exists y(\neg x = y \wedge Rxy \wedge Ryx)$ | True |
| 10. $\exists x\forall y(Rxy \leftrightarrow x = y)$ | True |
| 11. $\exists x\forall y(Ryx \leftrightarrow x = y)$ | False |
| 12. $\exists x\exists y(\neg x = y \wedge Rxy \wedge \forall z(Rzx \leftrightarrow y = z))$ | True |

Basic rules for TFL

A. The following two ‘proofs’ are *incorrect*. Explain the mistakes they make.

1	$\neg L \rightarrow (A \wedge L)$	
2	$\neg L$	
3	A	$\rightarrow E$ 1, 2
4	L	
5	\perp	$\perp I$ 4, 2
6	A	$\perp E$ 5
7	A	TND 2–3, 4–6

$\rightarrow E$ on line 3 should yield ‘ $A \wedge L$ ’. ‘ A ’ could then be obtained by $\wedge E$.
 $\perp I$ on line 5 illicitly refers to a line from a closed subproof (line 2).

1	$A \wedge (B \wedge C)$	
2	$(B \vee C) \rightarrow D$	
3	B	$\wedge E$ 1
4	$B \vee C$	$\vee I$ 3
5	D	$\rightarrow E$ 4, 2

$\wedge E$ on line 3 should yield ‘ $B \wedge C$ ’. ‘ B ’ could then be obtained by $\wedge E$ again. The citation for line 5 is the wrong way round: it should be ‘ $\rightarrow E$ 2, 4’.

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs. Additionally, write down the argument that corresponds to each proof.

1	$P \wedge S$	
2	$S \rightarrow R$	
3	P	$\wedge E$ 1
4	S	$\wedge E$ 1
5	R	$\rightarrow E$ 2, 4
6	$R \vee E$	$\vee I$ 5

Corresponding argument:
 $P \wedge S, S \rightarrow R \therefore R \vee E$

1	$A \rightarrow D$	
2	$A \wedge B$	
3	A	$\wedge E$ 2
4	D	$\rightarrow E$ 1, 3
5	$D \vee E$	$\vee I$ 4
6	$(A \wedge B) \rightarrow (D \vee E)$	$\rightarrow I$ 2–5

Corresponding argument:
 $A \rightarrow D \therefore (A \wedge B) \rightarrow (D \vee E)$

1	$\neg L \rightarrow (J \vee L)$		
2	$\neg L$		
3	$J \vee L$	$\rightarrow E$ 1, 2	
4	J		
5	$J \wedge J$	$\wedge I$ 4, 4	
6	J	$\wedge E$ 5	
7	L		
8	\perp	$\perp I$ 7, 2	
9	J	$\perp E$ 8	
10	J	$\vee E$ 3, 4-6, 7-9	

Corresponding argument:
 $\neg L \rightarrow (J \vee L), \neg L \therefore J$

C. Give a proof for each of the following arguments:

1. $J \rightarrow \neg J \therefore \neg J$

1	$J \rightarrow \neg J$		
2	J		
3	$\neg J$	$\rightarrow E$ 1, 2	
4	\perp	$\perp I$ 2, 3	
5	$\neg J$	$\neg I$ 2-4	

2. $Q \rightarrow (Q \wedge \neg Q) \therefore \neg Q$

1	$Q \rightarrow (Q \wedge \neg Q)$		
2	Q		
3	$Q \wedge \neg Q$	$\rightarrow E$ 1, 2	
4	$\neg Q$	$\wedge E$ 3	
5	\perp	$\perp I$ 2, 4	
6	$\neg Q$	$\neg I$ 2-6	

3. $A \rightarrow (B \rightarrow C) \therefore (A \wedge B) \rightarrow C$

1	$A \rightarrow (B \rightarrow C)$		
2	$A \wedge B$		
3	A	$\wedge E$ 2	
4	$B \rightarrow C$	$\rightarrow E$ 1, 3	
5	B	$\wedge E$ 2	
6	C	$\rightarrow E$ 4, 5	
7	$(A \wedge B) \rightarrow C$	$\rightarrow I$ 2-6	

4. $K \wedge L \therefore K \leftrightarrow L$

1	$K \wedge L$	
2		
3		K
4		L
5		L
6		K
7	$K \leftrightarrow L$	$\leftrightarrow I$ 2-3, 4-5

5. $(C \wedge D) \vee E \therefore E \vee D$

1	$(C \wedge D) \vee E$	
2		
3		$C \wedge D$
4		D
5		$E \vee D$
6		E
7		$E \vee D$
8	$E \vee D$	$\vee E$ 1, 2-4, 5-6

6. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$

1	$A \leftrightarrow B$	
2	$B \leftrightarrow C$	
3		
4		A
5		B
6		C
7		C
8		B
9		A
10	$A \leftrightarrow C$	$\leftrightarrow I$ 3-5, 6-8

7. $\neg F \rightarrow G, F \rightarrow H \therefore G \vee H$

1	$\neg F \rightarrow G$	
2	$F \rightarrow H$	
3		
4		F
5		H
6		$G \vee H$
7		$\neg F$
8		G
9		$G \vee H$
10	$G \vee H$	TND 3-5, 6-8

8. $(Z \wedge K) \vee (K \wedge M), K \rightarrow D \therefore D$

1	$(Z \wedge K) \vee (K \wedge M)$	
2	$K \rightarrow D$	
3	$Z \wedge K$	
4	K	$\wedge E$ 3
5	$K \wedge M$	
6	K	$\wedge E$ 5
7	K	$\vee E$ 1, 3-4, 5-6
8	D	$\rightarrow E$ 2, 7

9. $P \wedge (Q \vee R), P \rightarrow \neg R \therefore Q \vee E$

1	$P \wedge (Q \vee R)$	
2	$P \rightarrow \neg R$	
3	P	$\wedge E$ 1
4	$\neg R$	$\rightarrow E$ 2, 3
5	$Q \vee R$	$\wedge E$ 1
6	Q	
7	$Q \vee E$	$\vee I$ 6
8	R	
9	\perp	$\perp I$ 8, 4
10	$Q \vee E$	$\perp E$ 9
11	$Q \vee E$	$\vee E$ 5, 6-7, 8-10

10. $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$

1	$S \leftrightarrow T$	
2	S	
3	T	$\leftrightarrow E$ 1, 2
4	$T \vee S$	$\vee I$ 3
5	$T \vee S$	
6	T	
7	S	$\leftrightarrow E$ 1, 6
8	S	
9	$S \wedge S$	$\wedge I$ 8, 8
10	S	$\wedge E$ 9
11	S	$\vee E$ 5, 6-7, 8-10
12	$S \leftrightarrow (T \vee S)$	$\leftrightarrow I$ 2-4, 5-11

11. $\neg(P \rightarrow Q) \therefore \neg Q$

1		$\neg(P \rightarrow Q)$	
2			
3			
4			
5			
6			
7			
8			

2		Q	
3			
4			
5			
6			
7			
8			

4			$Q \wedge Q$	$\wedge I$ 2, 2
5			Q	$\wedge E$ 4
6			$P \rightarrow Q$	$\rightarrow I$ 3-5
7			\perp	$\perp I$ 6, 1
8			$\neg Q$	$\neg I$ 2-7

12. $\neg(P \rightarrow Q) \therefore P$

1		$\neg(P \rightarrow Q)$	
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

2		P	
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

3			$P \wedge P$	$\wedge I$ 2, 2
4			P	$\wedge E$ 3
5			$\neg P$	
6				
7				
8				
9				
10				
11				
12				

7			\perp	$\perp I$ 6, 5
8			Q	$\perp E$ 7
9			$P \rightarrow Q$	$\rightarrow I$ 6-8
10			\perp	$\perp I$ 9, 1
11			P	$\perp E$ 10
12			P	TND 2-4, 5-11

Additional rules for TFL

A. The following proofs are missing their citations (rule and line numbers). Add them wherever they are required:

1	$W \rightarrow \neg B$	
2	$A \wedge W$	
3	$B \vee (J \wedge K)$	
4	W	$\wedge E$ 2
5	$\neg B$	$\rightarrow E$ 1, 4
6	$J \wedge K$	DS 3, 5
7	K	$\wedge E$ 6

1	$L \leftrightarrow \neg O$	
2	$L \vee \neg O$	
3	$\neg L$	
4	$\neg O$	DS 2, 3
5	L	$\leftrightarrow E$ 1, 4
6	\perp	$\perp I$ 5, 3
7	$\neg\neg L$	$\neg I$ 3-6
8	L	DNE 7

1	$Z \rightarrow (C \wedge \neg N)$	
2	$\neg Z \rightarrow (N \wedge \neg C)$	
3	$\neg(N \vee C)$	
4	$\neg N \wedge \neg C$	DeM 3
5	$\neg N$	$\wedge E$ 4
6	$\neg C$	$\wedge E$ 4
7	Z	
8	$C \wedge \neg N$	$\rightarrow E$ 1, 7
9	C	$\wedge E$ 8
10	\perp	$\perp I$ 9, 6
11	$\neg Z$	$\neg I$ 7-10
12	$N \wedge \neg C$	$\rightarrow E$ 2, 11
13	N	$\wedge E$ 12
14	\perp	$\perp I$ 13, 5
15	$\neg\neg(N \vee C)$	$\neg I$ 3-14
16	$N \vee C$	DNE 15

B. Give a proof for each of these arguments:

- $E \vee F, F \vee G, \neg F \therefore E \wedge G$

1	$E \vee F$	
2	$F \vee G$	
3	$\neg F$	
4	E	DS 1, 3
5	G	DS 2, 3
6	$E \wedge G$	$\wedge I$ 4, 5

2. $M \vee (N \rightarrow M) \therefore \neg M \rightarrow \neg N$

1		$M \vee (N \rightarrow M)$	
2			
3		$\neg M$	
3			
3			$N \rightarrow M$ DS 1, 2
4			$\neg N$ MT 3, 2
5		$\neg M \rightarrow \neg N$	→I 2-4

3. $(M \vee N) \wedge (O \vee P), N \rightarrow P, \neg P \therefore M \wedge O$

1		$(M \vee N) \wedge (O \vee P)$	
2		$N \rightarrow P$	
3		$\neg P$	
4		$\neg N$	MT 2, 3
5		$M \vee N$	∧E 1
6		M	DS 5, 4
7		$O \vee P$	∧E 1
8		O	DS 7, 3
9		$M \wedge O$	∧I 6, 8

4. $(X \wedge Y) \vee (X \wedge Z), \neg(X \wedge D), D \vee M \therefore M$

1		$(X \wedge Y) \vee (X \wedge Z)$	
2		$\neg(X \wedge D)$	
3		$D \vee M$	
4			
4			
5			$X \wedge Y$
5			X ∧E 4
6			
6			
6			$X \wedge Z$
7			X ∧E 6
8		X	∨E 1, 4-5, 6-7
9			
9			
9			D
10			$X \wedge D$ ∧I 8, 9
11			\perp ⊥I 10, 2
12		$\neg D$	¬I 9-11
13		M	DS 3, 12

Proof-theoretic concepts

28

A. Show that each of the following sentences is a theorem:

1. $O \rightarrow O$

1			O	
2			O	R 1
3			$O \rightarrow O$	\rightarrow I 1-2

2. $N \vee \neg N$

1			N	
2			$N \vee \neg N$	\vee I 1
3			$\neg N$	
4			$N \vee \neg N$	\vee I 3
5			$N \vee \neg N$	TND 1-2, 3-4

3. $J \leftrightarrow [J \vee (L \wedge \neg L)]$

1			J	
2			$J \vee (L \wedge \neg L)$	\vee I 1
3			$J \vee (L \wedge \neg L)$	
4				$L \wedge \neg L$
5				L
6				$\neg L$
7				\perp
8				$\neg(L \wedge \neg L)$
9				J
10			$J \leftrightarrow [J \vee (L \wedge \neg L)]$	\leftrightarrow I 1-2, 3-9

4. $((A \rightarrow B) \rightarrow A) \rightarrow A$

1	$(A \rightarrow B) \rightarrow A$	
2	$\neg A$	
3	$\neg(A \rightarrow B)$	MT 1, 2
4	A	
5	\perp	\perp I 4, 2
6	B	\perp E 5
7	$A \rightarrow B$	\rightarrow I 4-6
8	\perp	\perp I 7, 3
9	$\neg\neg A$	\neg I 2
10	A	DNE 9
11	$((A \rightarrow B) \rightarrow A) \rightarrow A$	\rightarrow I 1-10

B. Provide proofs to show each of the following:

1. $C \rightarrow (E \wedge G), \neg C \rightarrow G \vdash G$

1	$C \rightarrow (E \wedge G)$	
2	$\neg C \rightarrow G$	
3	C	
4	$E \wedge G$	\rightarrow E 1, 3
5	G	\wedge E 4
6	$\neg C$	
7	G	\rightarrow E 2, 6
8	G	TND 3-5, 6-7

2. $M \wedge (\neg N \rightarrow \neg M) \vdash (N \wedge M) \vee \neg M$

1	$M \wedge (\neg N \rightarrow \neg M)$	
2	M	\wedge E 1
3	$\neg N \rightarrow \neg M$	\wedge E 1
4	$\neg N$	
5	$\neg M$	\rightarrow E 3, 4
6	\perp	\perp I 2, 5
7	$\neg\neg N$	\neg I 4-6
8	N	DNE 7
9	$N \wedge M$	\wedge I 8, 2
10	$(N \wedge M) \vee \neg M$	\vee I 9

3. $(Z \wedge K) \leftrightarrow (Y \wedge M), D \wedge (D \rightarrow M) \vdash Y \rightarrow Z$

1	$(Z \wedge K) \leftrightarrow (Y \wedge M)$	
2	$D \wedge (D \rightarrow M)$	
3	D	$\wedge E$ 2
4	$D \rightarrow M$	$\wedge E$ 2
5	M	$\rightarrow E$ 4, 3
6	Y	
7	$Y \wedge M$	$\wedge I$ 6, 5
8	$Z \wedge K$	$\leftrightarrow E$ 1, 7
9	Z	$\wedge E$ 8
10	$Y \rightarrow Z$	$\rightarrow I$ 6-9

4. $(W \vee X) \vee (Y \vee Z), X \rightarrow Y, \neg Z \vdash W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	
2	$X \rightarrow Y$	
3	$\neg Z$	
4	$W \vee X$	
5	W	
6	$W \vee Y$	$\vee I$ 5
7	X	
8	Y	$\rightarrow E$ 2, 7
9	$W \vee Y$	$\vee I$ 8
10	$W \vee Y$	$\vee E$ 4, 5-6, 7-9
11	$Y \vee Z$	
12	Y	DS 11, 3
13	$W \vee Y$	$\vee I$ 12
14	$W \vee Y$	$\vee E$ 1, 4-10, 11-13

C. Show that each of the following pairs of sentences are provably equivalent:

1. $R \leftrightarrow E, E \leftrightarrow R$

1	$R \leftrightarrow E$	
2	E	
3	R	$\leftrightarrow E$ 1, 2
4	R	
5	E	$\leftrightarrow E$ 1, 4
6	$E \leftrightarrow R$	$\leftrightarrow I$ 2-3, 4-5

1	$E \leftrightarrow R$	
2	E	
3	R	$\leftrightarrow E$ 1, 2
4	R	
5	E	$\leftrightarrow E$ 1, 4
6	$R \leftrightarrow E$	$\leftrightarrow I$ 4-5, 2-3

2. $G, \neg\neg\neg\neg G$

1		G			
2			$\neg\neg\neg G$		
3				$\neg G$	DNE 2
4				\perp	\perp I 1, 3
5		$\neg\neg\neg\neg G$	\neg I 2-4		

1		$\neg\neg\neg\neg G$	
2		$\neg\neg G$	DNE 1
3		G	DNE 2

3. $T \rightarrow S, \neg S \rightarrow \neg T$

1		$T \rightarrow S$			
2			$\neg S$		
3				$\neg T$	MT 1, 2
4		$\neg S \rightarrow \neg T$	\rightarrow I 2-3		

1		$\neg S \rightarrow \neg T$				
2			T			
3				$\neg S$		
4					$\neg T$	\rightarrow E 1, 3
5					\perp	\perp I 2, 4
6			$\neg\neg S$	\neg I 3-5		
7		S	DNE 6			
8		$T \rightarrow S$	\rightarrow I 2-7			

4. $U \rightarrow I, \neg(U \wedge \neg I)$

1		$U \rightarrow I$			
2			$U \wedge \neg I$		
3				U	\wedge E 2
4				$\neg I$	\wedge E 2
5			I	\rightarrow E 1, 3	
6			\perp	\perp I 5, 4	
7		$\neg(U \wedge \neg I)$	\neg I 2-6		

1		$\neg(U \wedge \neg I)$				
2			U			
3				$\neg I$		
4					$U \wedge \neg I$	\wedge I 2, 3
5					\perp	\perp I 4, 1
6			$\neg\neg I$	\neg I 3-5		
7		I	DNE 6			
8		$U \rightarrow I$	\rightarrow I 2-7			

5. $\neg(C \rightarrow D), C \wedge \neg D$

1	$C \wedge \neg D$	
2	C	$\wedge E$ 1
3	$\neg D$	$\wedge E$ 1
4	$C \rightarrow D$	
5	D	$\rightarrow E$ 4, 2
6	\perp	$\perp I$ 5, 3
7	$\neg(C \rightarrow D)$	$\neg I$ 4-6

1	$\neg(C \rightarrow D)$	
2	D	
3	C	
4	D	R 2
5	$C \rightarrow D$	$\rightarrow I$ 3-4
6	\perp	$\perp I$ 5, 1
7	$\neg D$	$\neg I$ 2-6
8	$\neg C$	
9	C	
10	\perp	$\perp I$ 9, 8
11	D	$\perp E$ 10
12	$C \rightarrow D$	$\rightarrow I$ 9-11
13	\perp	$\perp I$ 12, 1
14	$\neg\neg C$	$\neg I$ 8-13
15	C	DNE 14
16	$C \wedge \neg D$	$\wedge I$ 15, 7

6. $\neg G \leftrightarrow H, \neg(G \leftrightarrow H)$

1	$\neg G \leftrightarrow H$	
2	$G \leftrightarrow H$	
3	G	
4	H	$\leftrightarrow E$ 2, 3
5	$\neg G$	$\leftrightarrow E$ 1, 4
6	\perp	$\perp I$ 3, 5
7	$\neg G$	
8	H	$\leftrightarrow E$ 1, 7
9	G	$\leftrightarrow E$ 2, 8
10	\perp	$\perp I$ 9, 7
11	\perp	TND 3-6, 7-10
12	$\neg(G \leftrightarrow H)$	$\neg I$ 2-11

1	$\neg(G \leftrightarrow H)$	
2	$\neg G$	
3	$\neg H$	
4	G	
5	\perp	$\perp I$ 4, 2
6	H	$\perp E$ 5
7	H	
8	\perp	$\perp I$ 7, 3
9	G	$\perp E$ 8
10	$G \leftrightarrow H$	$\leftrightarrow I$ 4-6, 7-9
11	\perp	$\perp I$ 10, 1
12	$\neg\neg H$	$\neg I$ 3-11
13	H	DNE 12
14	H	
15	G	
16	G	
17	H	R 14
18	H	
19	G	R 15
20	$G \leftrightarrow H$	$\leftrightarrow I$ 16-17, 18-19
21	\perp	$\perp I$ 20, 1
22	$\neg G$	$\neg I$ 15-21
23	$\neg G \leftrightarrow H$	$\leftrightarrow I$ 2-13, 14-22

D. If you know that $\mathcal{A} \vdash \mathcal{B}$, what can you say about $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$? What about $(\mathcal{A} \vee \mathcal{C}) \vdash \mathcal{B}$? Explain your answers.

If $\mathcal{A} \vdash \mathcal{B}$, then $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$. After all, if $\mathcal{A} \vdash \mathcal{B}$, then there is some proof with assumption \mathcal{A} that ends with \mathcal{B} , and no undischarged assumptions other than \mathcal{A} . Now, if we start a proof with assumption $(\mathcal{A} \wedge \mathcal{C})$, we can obtain \mathcal{A} by $\wedge E$. We can now copy and paste the original proof of \mathcal{B} from \mathcal{A} , adding 1 to every line number and line number citation. The result will be a proof of \mathcal{B} from assumption \mathcal{A} .

However, we cannot prove much from $(\mathcal{A} \vee \mathcal{C})$. After all, it might be impossible to prove \mathcal{B} from \mathcal{C} .

E. In this section, I claimed that it is just as hard to show that two sentences are not provably equivalent, as it is to show that a sentence is not a theorem. Why did I claim this? (*Hint*: think of a sentence that would be a theorem iff \mathcal{A} and \mathcal{B} were provably equivalent.)

Consider the sentence $\mathcal{A} \leftrightarrow \mathcal{B}$. Suppose we can show that this is a theorem.

So we can prove it, with no assumptions, in m lines, say. Then if we assume \mathcal{A} and copy and paste the proof of $\mathcal{A} \leftrightarrow \mathcal{B}$ (changing the line numbering), we will have a deduction of this shape:

$$\begin{array}{l|l} 1 & \mathcal{A} \\ m+1 & \mathcal{A} \leftrightarrow \mathcal{B} \\ m+2 & \mathcal{B} \quad \leftrightarrow E \ m+1, 1 \end{array}$$

This will show that $\mathcal{A} \vdash \mathcal{B}$. In exactly the same way, we can show that $\mathcal{B} \vdash \mathcal{A}$. So if we can show that $\mathcal{A} \leftrightarrow \mathcal{B}$ is a theorem, we can show that \mathcal{A} and \mathcal{B} are provably equivalent.

Conversely, suppose we can show that \mathcal{A} and \mathcal{B} are provably equivalent. Then we can prove \mathcal{B} from the assumption of \mathcal{A} in m lines, say, and prove \mathcal{A} from the assumption of \mathcal{B} in n lines, say. Copying and pasting these proofs together (changing the line numbering where appropriate), we obtain:

$$\begin{array}{l|l|l} 1 & & \mathcal{A} \\ m & & \mathcal{B} \\ m+1 & & \mathcal{B} \\ m+n & & \mathcal{A} \\ m+n+1 & \mathcal{A} \leftrightarrow \mathcal{B} & \leftrightarrow I \ 1-m, m+1-m+n \end{array}$$

Thus showing that $\mathcal{A} \leftrightarrow \mathcal{B}$ is a theorem.

There was nothing special about \mathcal{A} and \mathcal{B} in this. So what this shows is that the problem of showing that two sentences are provably equivalent is, essentially, the same problem as showing that a certain kind of sentence (a biconditional) is a theorem.

Derived rules

30

A. Provide proof schemes that justify the addition of the third and fourth De Morgan rules as derived rules.

Third rule:

m	$\neg\mathcal{A} \wedge \neg\mathcal{B}$					
k	$\neg\mathcal{A}$	$\wedge E\ m$				
$k+1$	$\neg\mathcal{B}$	$\wedge E\ m$				
$k+2$	$\mathcal{A} \vee \mathcal{B}$					
$k+3$	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">\mathcal{A}</td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">\perp</td> <td style="padding-left: 5px;">$\perp I\ k+3, k$</td> </tr> </table>	\mathcal{A}		\perp	$\perp I\ k+3, k$	
\mathcal{A}						
\perp	$\perp I\ k+3, k$					
$k+4$	\perp	$\perp I\ k+3, k$				
$k+5$	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">\mathcal{B}</td> <td></td> </tr> <tr> <td style="border-right: 1px solid black; padding-right: 5px;">\perp</td> <td style="padding-left: 5px;">$\perp I\ k+5, k+1$</td> </tr> </table>	\mathcal{B}		\perp	$\perp I\ k+5, k+1$	
\mathcal{B}						
\perp	$\perp I\ k+5, k+1$					
$k+6$	\perp	$\perp I\ k+5, k+1$				
$k+7$	\perp	$\vee E\ k+2, k+3-k+4, k+5-k+6$				
$k+8$	$\neg(\mathcal{A} \vee \mathcal{B})$	$\neg I\ k+2-k+7$				

Fourth rule:

m	$\neg(\mathcal{A} \vee \mathcal{B})$	
k	\mathcal{A}	
$k+1$	$\mathcal{A} \vee \mathcal{B}$	$\vee I\ k$
$k+2$	\perp	$\perp I\ k+1, m$
$k+3$	$\neg\mathcal{A}$	$\neg I\ k-k+2$
$k+4$	\mathcal{B}	
$k+5$	$\mathcal{A} \vee \mathcal{B}$	$\vee I\ k+4$
$k+6$	\perp	$\perp I\ k+5, m$
$k+7$	$\neg\mathcal{B}$	$\neg I\ k+4-k+6$
$k+8$	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	$\wedge I\ k+3, k+7$

Basic rules for FOL

A. The following two ‘proofs’ are *incorrect*. Explain why both are incorrect. Also, provide interpretations which would invalidate the fallacious argument forms the ‘proofs’ enshrine:

1	$\forall xRxx$	
2	Raa	$\forall E$ 1
3	$\forall yRay$	$\forall I$ 2
4	$\forall x\forall yRxy$	$\forall I$ 3

1	$\forall x\exists yRxy$					
2	$\exists yRay$	$\forall E$ 1				
3	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">Raa</td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$\exists xRxx$</td> <td style="padding-left: 10px;">$\exists I$ 3</td> </tr> </table>	Raa		$\exists xRxx$	$\exists I$ 3	
Raa						
$\exists xRxx$	$\exists I$ 3					
5	$\exists xRxx$	$\exists E$ 2, 3–4				

When using $\forall I$, you must replace *all* names with the new variable. So line 3 is bogus. As a counterinterpretation, consider the following:



The instantiating constant, ‘a’, occurs in the line (line 2) to which $\exists E$ is to be applied on line 5. So the use of $\exists E$ on line 5 is bogus. As a counterinterpretation, consider the following:



B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs.

1	$\forall x\exists y(Rxy \vee Ryx)$									
2	$\forall x\neg Rmx$									
3	$\exists y(Rmy \vee Rym)$	$\forall E$ 1								
4	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$Rma \vee Ram$</td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$\neg Rma$</td> <td style="padding-left: 10px;">$\forall E$ 2</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">Ram</td> <td style="padding-left: 10px;">DS 4, 5</td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$\exists xRxm$</td> <td style="padding-left: 10px;">$\exists I$ 6</td> </tr> </table>	$Rma \vee Ram$		$\neg Rma$	$\forall E$ 2	Ram	DS 4, 5	$\exists xRxm$	$\exists I$ 6	
$Rma \vee Ram$										
$\neg Rma$	$\forall E$ 2									
Ram	DS 4, 5									
$\exists xRxm$	$\exists I$ 6									
8	$\exists xRxm$	$\exists E$ 3, 4–7								

1	$\forall x(\exists yLxy \rightarrow \forall zLzx)$	
2	Lab	
3	$\exists yLay \rightarrow \forall zLza$	$\forall E$ 1
4	$\exists yLay$	$\exists I$ 2
5	$\forall zLza$	$\rightarrow E$ 3, 4
6	Lca	$\forall E$ 5
7	$\exists yLcy \rightarrow \forall zLzc$	$\forall E$ 1
8	$\exists yLcy$	$\exists I$ 6
9	$\forall zLzc$	$\rightarrow E$ 7, 8
10	Lcc	$\forall E$ 9
11	$\forall xLxx$	$\forall I$ 10

1	$\forall x(Jx \rightarrow Kx)$	
2	$\exists x\forall yLxy$	
3	$\forall xJx$	
4	$\forall yLay$	
5	Laa	$\forall E$ 4
6	Ja	$\forall E$ 3
7	$Ja \rightarrow Ka$	$\forall E$ 1
8	Ka	$\rightarrow E$ 7, 6
9	$Ka \wedge Laa$	$\wedge I$ 8, 5
10	$\exists x(Kx \wedge Lxx)$	$\exists I$ 9
11	$\exists x(Kx \wedge Lxx)$	$\exists E$ 2, 4–10

C. In §15 problem part A, we considered fifteen syllogistic figures of Aristotelian logic. Provide proofs for each of the argument forms. NB: You will find it *much* easier if you symbolise (for example) ‘No F is G’ as ‘ $\forall x(Fx \rightarrow \neg Gx)$ ’.

I shall prove the four Figure I syllogisms; the rest are *extremely* similar.

Barbara

1	$\forall x(Gx \rightarrow Fx)$	
2	$\forall x(Hx \rightarrow Gx)$	
3	$Ga \rightarrow Fa$	$\forall E$ 1
4	$Ha \rightarrow Ga$	$\forall E$ 2
5	Ha	
6	Ga	$\rightarrow E$ 4, 5
7	Fa	$\rightarrow E$ 3, 6
8	$Ha \rightarrow Fa$	$\rightarrow I$ 5–7
9	$\forall x(Hx \rightarrow Fx)$	$\forall I$ 8

Celerant is exactly as Barbara, replacing ‘ F ’ with ‘ $\neg F$ ’ throughout.

Ferio

1	$\forall x(Gx \rightarrow \neg Fx)$	
2	$\exists x(Hx \wedge Gx)$	
3	$Ha \wedge Ga$	
4	Ha	$\wedge E$ 3
5	Ga	$\wedge E$ 3
6	$Ga \rightarrow \neg Fa$	$\forall E$ 1
7	$\neg Fa$	$\rightarrow E$ 6, 5
8	$Ha \wedge \neg Fa$	$\wedge I$ 4, 7
9	$\exists x(Hx \wedge \neg Fx)$	$\exists I$ 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists E$ 2, 3–9

Darii is exactly as Ferio, replacing ‘ $\neg F$ ’ with ‘ F ’ throughout.

D. Aristotle and his successors identified other syllogistic forms which depended upon ‘existential import’. Symbolise each of the following argument forms in FOL and offer proofs.

- **Barbari.** Something is H. All G are F. All H are G. So: Some H is F
 $\exists xHx, \forall x(Gx \rightarrow Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge Fx)$

1	$\exists xHx$	
2	$\forall x(Gx \rightarrow Fx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	H_a	
5	$H_a \rightarrow G_a$	$\forall E$ 3
6	G_a	$\rightarrow E$ 5, 4
7	$G_a \rightarrow F_a$	$\forall E$ 2
8	F_a	$\rightarrow E$ 7, 6
9	$H_a \wedge F_a$	$\wedge I$ 4, 8
10	$\exists x(Hx \wedge Fx)$	$\exists I$ 9
11	$\exists x(Hx \wedge Fx)$	$\exists E$ 1, 4–10

- **Celarent.** Something is H. No G are F. All H are G. So: Some H is not F

$\exists xHx, \forall x(Gx \rightarrow \neg Fx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$

Proof is exactly as for Barbari, replacing ' F ' with ' $\neg F$ ' throughout.

- **Cesaro.** Something is H. No F are G. All H are G. So: Some H is not F.
 $\exists xHx, \forall x(Fx \rightarrow \neg Gx), \forall x(Hx \rightarrow Gx) \therefore \exists x(Hx \wedge \neg Fx)$

1	$\exists xHx$	
2	$\forall x(Fx \rightarrow \neg Gx)$	
3	$\forall x(Hx \rightarrow Gx)$	
4	H_a	
5	$H_a \rightarrow G_a$	$\forall E$ 3
6	G_a	$\rightarrow E$ 5, 4
7	$F_a \rightarrow \neg G_a$	$\forall E$ 2
8	F_a	
9	$\neg G_a$	$\rightarrow E$ 7, 8
10	\perp	$\perp I$ 6, 9
11	$\neg F_a$	$\neg I$ 8–10
12	$H_a \wedge \neg F_a$	$\wedge I$ 4, 11
13	$\exists x(Hx \wedge \neg Fx)$	$\exists I$ 12
14	$\exists x(Hx \wedge \neg Fx)$	$\exists E$ 1, 4–13

- **Camestros.** Something is H. All F are G. No H are G. So: Some H is not F.

$$\exists xHx, \forall x(Fx \rightarrow Gx), \forall x(Hx \rightarrow \neg Gx) \therefore \exists x(Hx \wedge \neg Fx)$$

1	$\exists xHx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Hx \rightarrow \neg Gx)$	
4	Hx	
5	$Hx \rightarrow \neg Gx$	$\forall E$ 3
6	$\neg Gx$	$\rightarrow E$ 5, 4
7	$Fx \rightarrow Gx$	$\forall E$ 2
8	$\neg Fx$	MT 7, 6
9	$Hx \wedge \neg Fx$	$\wedge I$ 4, 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists I$ 9
11	$\exists x(Hx \wedge \neg Fx)$	$\exists E$ 1, 4–10

- **Felapton.** Something is G. No G are F. All G are H. So: Some H is not F.

$$\exists xGx, \forall x(Gx \rightarrow \neg Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$$

1	$\exists xGx$	
2	$\forall x(Gx \rightarrow \neg Fx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	Gx	
5	$Gx \rightarrow Hx$	$\forall E$ 3
6	Hx	$\rightarrow E$ 5, 4
7	$Gx \rightarrow \neg Fx$	$\forall E$ 2
8	$\neg Fx$	$\rightarrow E$ 7, 4
9	$Hx \wedge \neg Fx$	$\wedge I$ 6, 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists I$ 9
11	$\exists x(Hx \wedge \neg Fx)$	$\exists E$ 1, 4–10

- **Darapti.** Something is G. All G are F. All G are H. So: Some H is F.

$$\exists xGx, \forall x(Gx \rightarrow Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$$

Proof is exactly as for Felapton, replacing ‘ $\neg F$ ’ with ‘ F ’ throughout.

- **Calemos.** Something is H. All F are G. No G are H. So: Some H is not F.

$$\exists xHx, \forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow \neg Hx) \therefore \exists x(Hx \wedge \neg Fx)$$

1	$\exists xHx$																											
2	$\forall x(Fx \rightarrow Gx)$																											
3	$\forall x(Gx \rightarrow \neg Hx)$																											
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- **Fesapo.** Something is G. No F is G. All G are H. So: Some H is not F.

$$\exists xGx, \forall x(Fx \rightarrow \neg Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$$

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- **Bamalip.** Something is F. All F are G. All G are H. So: Some H are F.
 $\exists xFx, \forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$

1	$\exists xFx$	
2	$\forall x(Fx \rightarrow Gx)$	
3	$\forall x(Gx \rightarrow Hx)$	
4	Fa	
5	$Fa \rightarrow Ga$	$\forall E$ 2
6	Ga	$\rightarrow E$ 5, 4
7	$Ga \rightarrow Ha$	$\forall E$ 3
8	Ha	$\rightarrow E$ 7, 6
9	$Ha \wedge Fa$	$\wedge I$ 8, 4
10	$\exists x(Hx \wedge Fx)$	$\exists I$ 9
11	$\exists x(Hx \wedge Fx)$	$\exists E$ 1, 4–10

E. Provide a proof of each claim.

1. $\vdash \forall xFx \vee \neg\forall xFx$

1	$\forall xFx$	
2	$\forall xFx \vee \neg\forall xFx$	$\vee I$ 1
3	$\neg\forall xFx$	
4	$\forall xFx \vee \neg\forall xFx$	$\vee I$ 3
5	$\forall xFx \vee \neg\forall xFx$	TND 1–2, 3–4

2. $\vdash \forall z(Pz \vee \neg Pz)$

1	Pa	
2	$Pa \vee \neg Pa$	$\vee I$ 1
3	$\neg Pa$	
4	$Pa \vee \neg Pa$	$\vee I$ 3
5	$Pa \vee \neg Pa$	TND 1–2, 3–4
6	$\forall x(Px \vee \neg Px)$	$\forall I$ 5

3. $\forall x(Ax \rightarrow Bx), \exists xAx \vdash \exists xBx$

1	$\forall x(Ax \rightarrow Bx)$	
2	$\exists xAx$	
3	Aa	
4	$Aa \rightarrow Ba$	$\forall E$ 1
5	Ba	$\rightarrow E$ 4, 3
6	$\exists xBx$	$\exists I$ 5
7	$\exists xBx$	$\exists E$ 2, 3–6

4. $\forall x(Mx \leftrightarrow Nx), Ma \wedge \exists xRxa \vdash \exists xNx$
- | | | |
|---|------------------------------------|--------------------------|
| 1 | $\forall x(Mx \leftrightarrow Nx)$ | |
| 2 | $Ma \wedge \exists xRxa$ | |
| | | |
| 3 | Ma | $\wedge E$ 2 |
| 4 | $Ma \leftrightarrow Na$ | $\forall E$ 1 |
| 5 | Na | $\leftrightarrow E$ 4, 3 |
| 6 | $\exists xNx$ | $\exists I$ 5 |
5. $\forall x\forall yGxy \vdash \exists xGxx$
- | | | |
|---|-------------------------|---------------|
| 1 | $\forall x\forall yGxy$ | |
| | | |
| 2 | $\forall yGay$ | $\forall E$ 1 |
| 3 | Gaa | $\forall E$ 2 |
| 4 | $\exists xGxx$ | $\exists I$ 3 |
6. $\vdash \forall xRxx \rightarrow \exists x\exists yRxy$
- | | | |
|---|--|---------------------|
| 1 | $\forall xRxx$ | |
| | | |
| 2 | Raa | $\forall E$ 1 |
| 3 | $\exists yRay$ | $\exists I$ 2 |
| 4 | $\exists x\exists yRxy$ | $\exists I$ 3 |
| 5 | $\forall xRxx \rightarrow \exists x\exists yRxy$ | $\rightarrow I$ 1-4 |
7. $\vdash \forall y\exists x(Qy \rightarrow Qx)$
- | | | |
|---|---|---------------------|
| 1 | Qa | |
| | | |
| 2 | Qa | R 1 |
| 3 | $Qa \rightarrow Qa$ | $\rightarrow I$ 1-2 |
| 4 | $\exists x(Qa \rightarrow Qx)$ | $\exists I$ 3 |
| 5 | $\forall y\exists x(Qy \rightarrow Qx)$ | $\forall I$ 4 |
8. $Na \rightarrow \forall x(Mx \leftrightarrow Ma), Ma, \neg Mb \vdash \neg Na$
- | | | |
|---|---|--------------------------|
| 1 | $Na \rightarrow \forall x(Mx \leftrightarrow Ma)$ | |
| 2 | Ma | |
| 3 | $\neg Mb$ | |
| | | |
| 4 | Na | |
| | | |
| 5 | $\forall x(Mx \leftrightarrow Ma)$ | $\rightarrow E$ 1, 4 |
| 6 | $Mb \leftrightarrow Ma$ | $\forall E$ 5 |
| 7 | Mb | $\leftrightarrow E$ 6, 2 |
| 8 | \perp | $\perp I$ 7, 3 |
| 9 | $\neg Na$ | $\neg I$ 4-8 |

9. $\forall x\forall y(Gxy \rightarrow Gyx) \vdash \forall x\forall y(Gxy \leftrightarrow Gyx)$

1	$\forall x\forall y(Gxy \rightarrow Gyx)$	
2	Gab	
3	$\forall y(Gay \rightarrow Gya)$	$\forall E$ 1
4	$Gab \rightarrow Gba$	$\forall E$ 3
5	Gba	$\rightarrow E$ 4, 2
6	Gba	
7	$\forall y(Gby \rightarrow Gyb)$	$\forall E$ 1
8	$Gba \rightarrow Gab$	$\forall E$ 7
9	Gab	$\rightarrow E$ 8, 6
10	$Gab \leftrightarrow Gba$	$\leftrightarrow I$ 2–5, 6–9
11	$\forall y(Gay \leftrightarrow Gya)$	$\forall I$ 10
12	$\forall x\forall y(Gxy \leftrightarrow Gyx)$	$\forall I$ 11

10. $\forall x(\neg Mx \vee Ljx), \forall x(Bx \rightarrow Ljx), \forall x(Mx \vee Bx) \vdash \forall xLjx$

1	$\forall x(\neg Mx \vee Ljx)$	
2	$\forall x(Bx \rightarrow Ljx)$	
3	$\forall x(Mx \vee Bx)$	
4	$\neg Ma \vee Lja$	$\forall E$ 1
5	$Ba \rightarrow Lja$	$\forall E$ 2
6	$Ma \vee Ba$	$\forall E$ 3
7	$\neg Ma$	
8	Ba	DS 6, 7
9	Lja	$\rightarrow E$ 5, 8
10	Lja	
11	Lja	R 10
12	Lja	$\forall E$ 4, 7–9, 10–11
13	$\forall xLjx$	$\forall I$ 12

F. Write a symbolisation key for the following argument, symbolise it, and prove it:

There is someone who likes everyone who likes everyone that she likes. Therefore, there is someone who likes herself.

Symbolisation key:

domain: all people

Lxy : _____ x likes _____ y

$\exists x\forall y(\forall z(Lxz \rightarrow Lyz) \rightarrow Lxy) \therefore \exists xLxx$

1	$\exists x \forall y (\forall z (Lxz \rightarrow Lyz) \rightarrow Lxy)$	
2	$\forall y (\forall z (Laz \rightarrow Lyz) \rightarrow Lay)$	
3	$\forall z (Laz \rightarrow Laz) \rightarrow Laa$	$\forall E$ 2
4	Lac	
5	Lac	R 4
6	$Lac \rightarrow Lac$	$\rightarrow I$ 4–5
7	$\forall z (Laz \rightarrow Laz)$	$\forall I$ 6
8	Laa	$\rightarrow E$ 3, 7
9	$\exists x Lxx$	$\exists I$ 8
10	$\exists x Lxx$	$\exists E$ 1, 2–9

G. For each of the following pairs of sentences: If they are provably equivalent, give proofs to show this. If they are not, construct an interpretation to show that they are not logically equivalent.

1. $\forall x Px \rightarrow Qc, \forall x (Px \rightarrow Qc)$ Not logically equivalent
 Counter-interpretation: let the domain be the numbers 1 and 2. Let ‘ c ’ name 1. Let ‘ Px ’ be true of and only of 1. Let ‘ Qx ’ be true of, and only of, 2.
2. $\forall x \forall y \forall z Bxyz, \forall x Bxxx$ Not logically equivalent
 Counter-interpretation: let the domain be the numbers 1 and 2. Let ‘ $Bxyz$ ’ be true of, and only of, $\langle 1,1,1 \rangle$ and $\langle 2,2,2 \rangle$.
3. $\forall x \forall y Dxy, \forall y \forall x Dxy$ Provably equivalent

<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">1</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall x \forall y Dxy$ </td> <td></td> </tr> <tr> <td style="text-align: right; vertical-align: top;">2</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall y Day$ </td> <td style="vertical-align: top;">$\forall E$ 1</td> </tr> <tr> <td style="text-align: right; vertical-align: top;">3</td> <td style="border-left: 1px solid black; padding-left: 5px;"> Dab </td> <td style="vertical-align: top;">$\forall E$ 2</td> </tr> <tr> <td style="text-align: right; vertical-align: top;">4</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall x Dxb$ </td> <td style="vertical-align: top;">$\forall I$ 3</td> </tr> <tr> <td style="text-align: right; vertical-align: top;">5</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall y \forall x Dxy$ </td> <td style="vertical-align: top;">$\forall I$ 4</td> </tr> </table>	1	$\forall x \forall y Dxy$		2	$\forall y Day$	$\forall E$ 1	3	Dab	$\forall E$ 2	4	$\forall x Dxb$	$\forall I$ 3	5	$\forall y \forall x Dxy$	$\forall I$ 4	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="width: 5%; text-align: right; vertical-align: top;">1</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall y \forall x Dxy$ </td> <td></td> </tr> <tr> <td style="text-align: right; vertical-align: top;">2</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall x Dxa$ </td> <td style="vertical-align: top;">$\forall E$ 1</td> </tr> <tr> <td style="text-align: right; vertical-align: top;">3</td> <td style="border-left: 1px solid black; padding-left: 5px;"> Dba </td> <td style="vertical-align: top;">$\forall E$ 2</td> </tr> <tr> <td style="text-align: right; vertical-align: top;">4</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall y Dby$ </td> <td style="vertical-align: top;">$\forall I$ 3</td> </tr> <tr> <td style="text-align: right; vertical-align: top;">5</td> <td style="border-left: 1px solid black; padding-left: 5px;"> $\forall x \forall y Dxy$ </td> <td style="vertical-align: top;">$\forall I$ 4</td> </tr> </table>	1	$\forall y \forall x Dxy$		2	$\forall x Dxa$	$\forall E$ 1	3	Dba	$\forall E$ 2	4	$\forall y Dby$	$\forall I$ 3	5	$\forall x \forall y Dxy$	$\forall I$ 4
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4	$\forall y Dby$	$\forall I$ 3																													
5	$\forall x \forall y Dxy$	$\forall I$ 4																													

4. $\exists x \forall y Dxy, \forall y \exists x Dxy$ Not logically equivalent
 Counter-interpretation: let the domain be the numbers 1 and 2. Let ‘ Dxy ’ hold of and only of $\langle 1,2 \rangle$ and $\langle 2,1 \rangle$. This is depicted thus:



5. $\forall x (Rca \leftrightarrow Rxa), Rca \leftrightarrow \forall x Rxa$ Not logically equivalent
 Counter-interpretation, consider the following diagram, allowing ‘ a ’ to name 1 and ‘ c ’ to name 2:



H. For each of the following arguments: If it is valid in FOL, give a proof. If it is invalid, construct an interpretation to show that it is invalid.

1. $\exists y \forall x Rxy \therefore \forall x \exists y Rxy$ Valid
- | | | |
|---|---------------------------|--------------------|
| 1 | $\exists y \forall x Rxy$ | |
| 2 | $\forall x Rxa$ | |
| 3 | Rba | $\forall E$ 2 |
| 4 | $\exists y Rby$ | $\exists I$ 3 |
| 5 | $\exists y Rby$ | $\exists E$ 1, 2-4 |
| 6 | $\forall x \exists y Rxy$ | $\forall I$ 5 |
2. $\exists x (Px \wedge \neg Qx) \therefore \forall x (Px \rightarrow \neg Qx)$ Not valid
Counter interpretation: let the domain be the numbers 1 and 2. Let ‘ Px ’ be true of everything in the domain. Let ‘ Qx ’ be true of, and only of, 2.
3. $\forall x (Sx \rightarrow Ta), Sd \therefore Ta$ Valid
- | | | |
|---|---------------------------------|----------------------|
| 1 | $\forall x (Sx \rightarrow Ta)$ | |
| 2 | Sd | |
| 3 | $Sd \rightarrow Ta$ | $\forall E$ 1 |
| 4 | Ta | $\rightarrow E$ 3, 2 |
4. $\forall x (Ax \rightarrow Bx), \forall x (Bx \rightarrow Cx) \therefore \forall x (Ax \rightarrow Cx)$ Valid
- | | | |
|---|---------------------------------|----------------------|
| 1 | $\forall x (Ax \rightarrow Bx)$ | |
| 2 | $\forall x (Bx \rightarrow Cx)$ | |
| 3 | $Aa \rightarrow Ba$ | $\forall E$ 1 |
| 4 | $Ba \rightarrow Ca$ | $\forall E$ 2 |
| 5 | Aa | |
| 6 | Ba | $\rightarrow E$ 3, 5 |
| 7 | Ca | $\rightarrow E$ 4, 6 |
| 8 | $Aa \rightarrow Ca$ | $\rightarrow I$ 5-7 |
| 9 | $\forall x (Ax \rightarrow Cx)$ | $\forall I$ 8 |
5. $\exists x (Dx \vee Ex), \forall x (Dx \rightarrow Fx) \therefore \exists x (Dx \wedge Fx)$ Invalid
Counter-interpretation: let the domain be the number 1. Let ‘ Dx ’ hold of nothing. Let both ‘ Ex ’ and ‘ Fx ’ hold of everything.
6. $\forall x \forall y (Rxy \vee Ryx) \therefore Rjj$ Valid
- | | | |
|---|--------------------------------------|----------------------|
| 1 | $\forall x \forall y (Rxy \vee Ryx)$ | |
| 2 | $\forall y (Rjy \vee Ryj)$ | $\forall E$ 1 |
| 3 | $Rjj \vee Rjj$ | $\forall E$ 2 |
| 4 | Rjj | |
| 5 | Rjj | R 4 |
| 6 | Rjj | |
| 7 | Rjj | R 6 |
| 8 | Rjj | $\vee E$ 3, 4-5, 6-7 |

7. $\exists x \exists y (Rxy \vee Ryx) \therefore Rjj$ Invalid
Counter-interpretation: consider the following diagram, allowing 'j' to name 2.



8. $\forall x Px \rightarrow \forall x Qx, \exists x \neg Px \therefore \exists x \neg Qx$ Invalid
Counter-interpretation: let the domain be the number 1. Let 'Px' be true of nothing. Let 'Qx' be true of everything.

Conversion of quantifiers

32

A. Show that the following are jointly contrary:

1. $Sa \rightarrow Tm, Tm \rightarrow Sa, Tm \wedge \neg Sa$

1		$Sa \rightarrow Tm$	
2		$Tm \rightarrow Sa$	
3		$Tm \wedge \neg Sa$	
4		Tm	$\wedge E$ 3
5		$\neg Sa$	$\wedge E$ 3
6		Sa	$\rightarrow E$ 2, 4
7		\perp	$\perp I$ 5, 6

2. $\neg \exists x Rxa, \forall x \forall y Ryx$

1		$\neg \exists x Rxa$	
2		$\forall x \forall y Ryx$	
3		$\forall x \neg Rxa$	CQ 1
4		$\neg Rba$	$\forall E$ 3
5		$\forall y Rya$	$\forall E$ 2
6		Rba	$\forall E$ 5
7		\perp	$\perp I$ 6, 4

3. $\neg \exists x \exists y Lxy, Laa$

1		$\neg \exists x \exists y Lxy$	
2		Laa	
3		$\forall x \neg \exists y Lxy$	CQ 1
4		$\neg \exists y Lay$	$\forall E$ 3
5		$\forall y \neg Lay$	CQ 4
6		$\neg Laa$	$\forall E$ 5
7		\perp	$\perp I$ 2, 6

4. $\forall x(Px \rightarrow Qx), \forall z(Pz \rightarrow Rz), \forall yPy, \neg Qa \wedge \neg Rb$

1	$\forall x(Px \rightarrow Qx)$	
2	$\forall z(Pz \rightarrow Rz)$	
3	$\forall yPy$	
4	$\neg Qa \wedge \neg Rb$	
	<hr style="width: 100%;"/>	
5	$\neg Qa$	$\wedge E$ 4
6	$Pa \rightarrow Qa$	$\forall E$ 1
7	$\neg Pa$	MT 6, 5
8	Pa	$\forall E$ 3
9	\perp	$\perp I$ 8, 7

B. Show that each pair of sentences is provably equivalent:

1. $\forall x(Ax \rightarrow \neg Bx), \neg \exists x(Ax \wedge Bx)$

1	$\forall x(Ax \rightarrow \neg Bx)$		1	$\neg \exists x(Ax \wedge Bx)$	
	<hr style="width: 100%;"/>			$\forall x \neg(Ax \wedge Bx)$	CQ 1
2	$\exists x(Ax \wedge Bx)$		2	$\neg(Aa \wedge Ba)$	$\forall E$ 2
	<hr style="width: 100%;"/>			<hr style="width: 100%;"/>	
3	$Aa \wedge Ba$		3	Aa	
	<hr style="width: 100%;"/>			<hr style="width: 100%;"/>	
4	Aa	$\wedge E$ 3	4	Ba	
5	Ba	$\wedge E$ 3	5	$Aa \wedge Ba$	$\wedge I$ 4, 5
6	$Aa \rightarrow \neg Ba$	$\forall E$ 1	6	\perp	$\perp I$ 6, 3
7	$\neg Ba$	$\rightarrow E$ 6, 4	7	$\neg Ba$	$\neg I$ 5-7
8	\perp	$\perp I$ 5, 7	8	$Aa \rightarrow \neg Ba$	$\rightarrow I$ 4-8
9	\perp	$\exists E$ 2, 3-8	9	$\forall x(Ax \rightarrow \neg Bx)$	$\forall I$ 9
10	$\neg \exists x(Ax \wedge Bx)$	$\neg I$ 2-9			

2. $\forall x(\neg Ax \rightarrow Bd), \forall xAx \vee Bd$

1	$\forall x(\neg Ax \rightarrow Bd)$		1	$\forall xAx \vee Bd$	
	<hr style="width: 100%;"/>			<hr style="width: 100%;"/>	
2	$\neg Aa \rightarrow Bd$	$\forall E$ 1	2	$\neg Aa$	
	<hr style="width: 100%;"/>			<hr style="width: 100%;"/>	
3	Bd		3	$\forall xAx$	
	<hr style="width: 100%;"/>			<hr style="width: 100%;"/>	
4	$\forall xAx \vee Bd$	$\vee I$ 6	4	Aa	$\forall E$ 3
5	$\neg Bd$		5	\perp	$\perp I$ 4, 2
	<hr style="width: 100%;"/>			<hr style="width: 100%;"/>	
6	$\neg \neg Aa$	MT 2, 5	6	$\neg \forall xAx$	$\neg I$ 3-5
7	Aa	DNE 6	7	Bd	DS 1, 6
8	$\forall xAx$	$\forall E$ 7	8	$\neg Aa \rightarrow Bd$	$\rightarrow I$ 2-7
9	$\forall xAx \vee Bd$	$\vee I$ 8	9	$\forall x(Ax \rightarrow Bd)$	$\forall I$ 8
10	$\forall xAx \vee Bd$	TND 3-4, 5-9			

C. In §15, I considered what happens when we move quantifiers ‘across’ various logical operators. Show that each pair of sentences is provably equivalent:

1. $\forall x(Fx \wedge Ga), \forall xFx \wedge Ga$

1	$\forall x(Fx \wedge Ga)$	
2	$Fb \wedge Ga$	$\forall E$ 1
3	Fb	$\wedge E$ 2
4	Ga	$\wedge E$ 2
5	$\forall xFx$	$\forall I$ 3
6	$\forall xFx \wedge Ga$	$\wedge I$ 5, 4

1	$\forall xFx \wedge Ga$	
2	$\forall xFx$	$\wedge E$ 1
3	Ga	$\wedge E$ 1
4	Fb	$\forall E$ 2
5	$Fb \wedge Ga$	$\wedge I$ 4, 3
6	$\forall x(Fx \wedge Ga)$	$\forall I$ 5

2. $\exists x(Fx \vee Ga), \exists xFx \vee Ga$

1	$\exists x(Fx \vee Ga)$	
2	$Fb \vee Ga$	
3	Fb	
4	$\exists xFx$	$\exists I$ 3
5	$\exists xFx \vee Ga$	$\vee I$ 4
6	Ga	
7	$\exists xFx \vee Ga$	$\vee I$ 6
8	$\exists xFx \vee Ga$	$\vee E$ 2, 3–5, 6–7
9	$\exists xFx \vee Ga$	$\exists E$ 1, 2–8

1	$\exists xFx \vee Ga$	
2	$\exists xFx$	
3	Fb	
4	$Fb \vee Ga$	$\vee I$ 3
5	$\exists x(Fx \vee Ga)$	$\exists I$ 4
6	$\exists x(Fx \vee Ga)$	$\exists E$ 2, 3–5
7	Ga	
8	$Fb \vee Ga$	$\vee I$ 7
9	$\exists x(Fx \vee Ga)$	$\exists I$ 8
10	$\exists x(Fx \vee Ga)$	$\vee E$ 1, 2–6, 7–9

3. $\forall x(Ga \rightarrow Fx), Ga \rightarrow \forall xFx$

1	$\forall x(Ga \rightarrow Fx)$	
2	$Ga \rightarrow Fb$	$\forall E$ 1
3	Ga	
4	Fb	$\rightarrow E$ 2, 3
5	$\forall xFx$	$\forall I$ 4
6	$Ga \rightarrow \forall xFx$	$\rightarrow I$ 3–5

1	$Ga \rightarrow \forall xFx$	
2	Ga	
3	$\forall xFx$	$\rightarrow E$ 1, 2
4	Fb	$\forall E$ 3
5	$Ga \rightarrow Fb$	$\rightarrow I$ 2–4
6	$\forall x(Ga \rightarrow Fx)$	$\forall I$ 5

4. $\forall x(Fx \rightarrow Ga), \exists xFx \rightarrow Ga$

1	$\forall x(Fx \rightarrow Ga)$	
2	$\exists xFx$	
3	Fb	
4	$Fb \rightarrow Ga$	$\forall E$ 1
5	Ga	$\rightarrow E$ 4, 3
6	Ga	$\exists E$ 2, 3-5
7	$\exists xFx \rightarrow Ga$	$\rightarrow I$ 2-6

1	$\exists xFx \rightarrow Ga$	
2	Fb	
3	$\exists xFx$	$\exists I$ 2
4	Ga	$\rightarrow E$ 1, 3
5	$Fb \rightarrow Ga$	$\rightarrow I$ 2-4
6	$\forall x(Fx \rightarrow Ga)$	$\forall I$ 5

5. $\exists x(Ga \rightarrow Fx), Ga \rightarrow \exists xFx$

1	$\exists x(Ga \rightarrow Fx)$	
2	Ga	
3	$Ga \rightarrow Fb$	
4	Fb	$\rightarrow E$ 3, 2
5	$\exists xFx$	$\exists I$ 4
6	$\exists xFx$	$\exists E$ 1, 3-5
7	$Ga \rightarrow \exists xFx$	$\rightarrow I$ 2-6

1	$Ga \rightarrow \exists xFx$	
2	Ga	
3	$\exists xFx$	
4	Fb	
5	Ga	
6	Fb	R 4
7	$Ga \rightarrow Fb$	$\rightarrow I$ 5-6
8	$\exists x(Ga \rightarrow Fx)$	$\exists I$ 7
9	$\exists x(Ga \rightarrow Fx)$	$\exists E$ 3, 4-8
10	$\neg Ga$	
11	Ga	
12	\perp	$\perp I$ 11, 10
13	Fb	$\perp E$ 12
14	$Ga \rightarrow Fb$	$\rightarrow E$ 11-13
15	$\exists x(Ga \rightarrow Fx)$	$\exists I$ 14
16	$\exists x(Ga \rightarrow Fx)$	TND 2-9, 10-15

6. $\exists x(Fx \rightarrow Ga), \forall xFx \rightarrow Ga$

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NB: the variable 'x' does not occur in 'Ga'.

When all the quantifiers occur at the beginning of a sentence, that sentence is said to be in *prenex normal form*. Together with the CQ rules, these equivalences are sometimes called *prenexing rules*, since they give us a means for putting any sentence into prenex normal form.

Rules for identity

33

A. Provide a proof of each claim.

1. $Pa \vee Qb, Qb \rightarrow b = c, \neg Pa \vdash Qc$

1		$Pa \vee Qb$	
2		$Qb \rightarrow b = c$	
3		$\neg Pa$	
4		Qb	DS 1, 3
5		$b = c$	$\rightarrow E$ 2, 4
6		Qc	$=E$ 5, 4

2. $m = n \vee n = o, An \vdash Am \vee Ao$

1		$m = n \vee n = o$	
2		An	
3		$m = n$	
4		Am	$=E$ 3, 2
5		$Am \vee Ao$	$\vee I$ 4
6		$n = o$	
7		Ao	$=E$ 6, 2
8		$Am \vee Ao$	$\vee I$ 7
9		$Am \vee Ao$	$\vee E$ 1, 3-5, 6-8

3. $\forall x x = m, Rma \vdash \exists x Rxx$

1		$\forall x x = m$	
2		Rma	
3		$a = m$	$\forall E$ 1
4		Raa	$=E$ 3, 2
5		$\exists x Rxx$	$\exists I$ 4

4. $\forall x \forall y (Rxy \rightarrow x = y) \vdash Rab \rightarrow Rba$

1	$\forall x \forall y (Rxy \rightarrow x = y)$	
2	Rab	
3	$\forall y (Ray \rightarrow a = y)$	$\forall E$ 1
4	$Rab \rightarrow a = b$	$\forall E$ 3
5	$a = b$	$\rightarrow E$ 4, 2
6	Raa	$=E$ 5, 2
7	Rba	$=E$ 5, 6
8	$Rab \rightarrow Rba$	$\rightarrow I$ 2-7

5. $\neg \exists x \neg x = m \vdash \forall x \forall y (Px \rightarrow Py)$

1	$\neg \exists x \neg x = m$	
2	$\forall x \neg \neg x = m$	CQ 1
3	$\neg \neg a = m$	$\forall E$ 2
4	$a = m$	DNE 3
5	$\neg \neg b = m$	$\forall E$ 2
6	$b = m$	DNE 5
7	Pa	
8	Pm	$=E$ 3, 7
9	Pb	$=E$ 5, 8
10	$Pa \rightarrow Pb$	$\rightarrow I$ 7-9
11	$\forall y (Pa \rightarrow Py)$	$\forall I$ 10
12	$\forall x \forall y (Px \rightarrow Py)$	$\forall I$ 11

6. $\exists x Jx, \exists x \neg Jx \vdash \exists x \exists y \neg x = y$

1	$\exists x Jx$	
2	$\exists x \neg Jx$	
3	Ja	
4	$\neg Jb$	
5	$a = b$	
6	Jb	$=E$ 5, 3
7	\perp	$\perp I$ 6, 4
8	$\neg a = b$	$\neg I$ 5-7
9	$\exists y \neg a = y$	$\exists I$ 8
10	$\exists x \exists y \neg x = y$	$\exists I$ 9
11	$\exists x \exists y \neg x = y$	$\exists E$ 2, 4-10
12	$\exists x \exists y \neg x = y$	$\exists E$ 1, 3-11

7. $\forall x(x = n \leftrightarrow Mx), \forall x(Ox \vee \neg Mx) \vdash On$

1	$\forall x(x = n \leftrightarrow Mx)$	
2	$\forall x(Ox \vee \neg Mx)$	
3	$n = n \leftrightarrow Mn$	$\forall E$ 1
4	$n = n$	$=I$
5	Mn	$\leftrightarrow E$ 3, 4
6	$On \vee \neg Mn$	$\forall E$ 2
7	$\neg On$	
8	$\neg Mn$	DS 6, 7
9	\perp	$\perp I$ 5, 8
10	$\neg\neg On$	$\neg I$ 7-9
11	On	DNE 10

8. $\exists xDx, \forall x(x = p \leftrightarrow Dx) \vdash Dp$

1	$\exists xDx$	
2	$\forall x(x = p \leftrightarrow Dx)$	
3	Dc	
4	$c = p \leftrightarrow Dc$	$\forall E$ 2
5	$c = p$	$\leftrightarrow E$ 4, 3
6	Dp	$=E$ 5, 3
7	Dp	$\exists E$ 1, 3-6

9. $\exists x[(Kx \wedge \forall y(Ky \rightarrow x = y)) \wedge Bx], Kd \vdash Bd$

1	$\exists x[(Kx \wedge \forall y(Ky \rightarrow x = y)) \wedge Bx]$	
2	Kd	
3	$(Ka \wedge \forall y(Ky \rightarrow a = y)) \wedge Ba$	
4	$Ka \wedge \forall y(Ky \rightarrow a = y)$	$\wedge E$ 3
5	Ka	$\wedge E$ 4
6	$\forall y(Ky \rightarrow a = y)$	$\wedge E$ 4
7	$Kd \rightarrow a = d$	$\forall E$ 6
8	$a = d$	$\rightarrow E$ 7, 2
9	Ba	$\wedge E$ 3
10	Bd	$=E$ 8, 9
11	Bd	$\exists E$ 1, 3-10

10. $\vdash Pa \rightarrow \forall x(Px \vee \neg x = a)$

1	Pa	
2	$b = a$	
3	Pb	$=E$ 2, 1
4	$Pb \vee \neg b = a$	$\vee I$ 3
5	$\neg b = a$	
6	$Pb \vee \neg b = a$	$\vee I$ 5
7	$Pb \vee \neg b = a$	TND 2–4, 5–6
8	$\forall x(Px \vee \neg x = a)$	$\forall I$ 7
9	$Pa \rightarrow \forall x(Px \vee \neg x = a)$	$\rightarrow I$ 1–8

B. Show that the following are provably equivalent:

- $\exists x([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n)$
- $Fn \wedge \forall y(Fy \rightarrow n = y)$

And hence that both have a decent claim to symbolise the English sentence ‘Nick is the F’.

In one direction:

1	$\exists x([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n)$	
2	$[Fa \wedge \forall y(Fy \rightarrow a = y)] \wedge a = n$	
3	$a = n$	$\wedge E$ 2
4	$Fa \wedge \forall y(Fy \rightarrow a = y)$	$\wedge E$ 2
5	Fa	$\wedge E$ 4
6	Fn	$=E$ 3, 5
7	$\forall y(Fy \rightarrow a = y)$	$\wedge E$ 4
8	$\forall y(Fy \rightarrow n = y)$	$=E$ 3, 7
9	$Fn \wedge \forall y(Fy \rightarrow n = y)$	$\wedge I$ 6, 8
10	$Fn \wedge \forall y(Fy \rightarrow n = y)$	$\exists E$ 1, 2–9

And now in the other:

1	$Fn \wedge \forall y(Fy \rightarrow n = y)$	
2	$n = n$	$=I$
3	$[Fn \wedge \forall y(Fy \rightarrow n = y)] \wedge n = n$	$\wedge I$ 1, 2
4	$\exists x([Fx \wedge \forall y(Fy \rightarrow x = y)] \wedge x = n)$	$\exists I$ 3

C. In §17, I claimed that the following are logically equivalent symbolisations of the English sentence ‘there is exactly one F’:

- $\exists xFx \wedge \forall x\forall y[(Fx \wedge Fy) \rightarrow x = y]$
- $\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$
- $\exists x\forall y(Fy \leftrightarrow x = y)$

Show that they are all provably equivalent. (*Hint*: to show that three claims are provably equivalent, it suffices to show that the first proves the second, the second proves the third and the third proves the first; think about why.)

It suffices to show that the first proves the second, the second proves the third and the third proves the first, for we can then show that any of them prove any others, just by chaining the proofs together (numbering lines, where necessary). Armed with this, we start on the first proof:

1	$\exists xFx \wedge \forall x\forall y[(Fx \wedge Fy) \rightarrow x = y]$	
2	$\exists xFx$	$\wedge E$ 1
3	$\forall x\forall y[(Fx \wedge Fy) \rightarrow x = y]$	$\wedge E$ 1
4	Fa	
5	$\forall y[(Fa \wedge Fy) \rightarrow a = y]$	$\forall E$ 3
6	$(Fa \wedge Fb) \rightarrow a = b$	$\forall E$ 5
7	Fb	
8	$Fa \wedge Fb$	$\wedge I$ 4, 7
9	$a = b$	$\rightarrow E$ 6, 8
10	$Fb \rightarrow a = b$	$\rightarrow I$ 7–9
11	$\forall y(Fy \rightarrow a = y)$	$\forall I$ 10
12	$Fa \wedge \forall y(Fy \rightarrow a = y)$	$\wedge I$ 4, 11
13	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	$\exists I$ 12
14	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	$\exists E$ 2, 4–13

Now for the second proof:

1	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	
2	$Fa \wedge \forall y(Fy \rightarrow a = y)$	
3	Fa	$\wedge E$ 2
4	$\forall y(Fy \rightarrow a = y)$	$\wedge E$ 2
5	Fb	
6	$Fb \rightarrow a = b$	$\forall E$ 4
7	$a = b$	$\rightarrow E$ 6, 5
8	$a = b$	
9	Fb	$=E$ 8, 3
10	$Fb \leftrightarrow a = b$	$\leftrightarrow I$ 5–7, 8–9
11	$\forall y(Fy \leftrightarrow a = y)$	$\forall I$ 10
12	$\exists x\forall y(Fy \leftrightarrow x = y)$	$\exists I$ 11
13	$\exists x\forall y(Fy \leftrightarrow x = y)$	$\exists E$ 1, 2–12

And finally, the third proof:

1	$\exists x \forall y (Fy \leftrightarrow x = y)$	
2	$\forall y (Fy \leftrightarrow a = y)$	
3	$Fa \leftrightarrow a = a$	$\forall E$ 2
4	$a = a$	$=I$
5	Fa	$\leftrightarrow E$ 3, 4
6	$\exists x Fx$	$\exists I$ 5
7	$Fb \wedge Fc$	
8	Fb	$\wedge E$ 7
9	$Fb \leftrightarrow a = b$	$\forall E$ 2
10	$a = b$	$\leftrightarrow E$ 9, 8
11	Fc	$\wedge E$ 7
12	$Fc \leftrightarrow a = c$	$\forall E$ 2
13	$a = c$	$\leftrightarrow E$ 12, 11
14	$b = c$	$=E$ 10, 13
15	$(Fb \wedge Fc) \rightarrow b = c$	$\rightarrow I$ 8–14
16	$\forall y [(Fb \wedge Fy) \rightarrow b = y]$	$\forall I$ 15
17	$\forall x \forall y [(Fx \wedge Fy) \rightarrow x = y]$	$\forall I$ 16
18	$\exists x Fx \wedge \forall x \forall y [(Fx \wedge Fy) \rightarrow x = y]$	$\wedge I$ 6, 17
19	$\exists x Fx \wedge \forall x \forall y [(Fx \wedge Fy) \rightarrow x = y]$	$\exists E$ 1, 2–18

D. Symbolise the following argument

There is exactly one F. There is exactly one G. Nothing is both F and G. So: there are exactly two things that are either F or G.

And offer a proof of it.

Here's the symbolisation, the proof will come over the page:

$\exists x [Fx \wedge \forall y (Fy \rightarrow x = y)]$,

$\exists x [Gx \wedge \forall y (Gy \rightarrow x = y)]$,

$\forall x (\neg Fx \vee \neg Gx) \therefore$

$\exists x \exists y [\neg x = y \wedge \forall z ((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$

1	$\exists x[Fx \wedge \forall y(Fy \rightarrow x = y)]$	
2	$\exists x[Gx \wedge \forall y(Gy \rightarrow x = y)]$	
3	$\forall x(\neg Fx \vee \neg Gx)$	
4	<u>$Fa \wedge \forall y(Fy \rightarrow a = y)$</u>	
5	Fa	$\wedge E$ 4
6	$\forall y(Fy \rightarrow a = y)$	$\wedge E$ 4
7	$\neg Fa \vee \neg Ga$	$\vee E$ 3
8	$\neg Ga$	DS 7, 5
9	<u>$Gb \wedge \forall y(Gy \rightarrow b = y)$</u>	
10	Gb	$\wedge E$ 9
11	$\forall y(Gy \rightarrow b = y)$	$\wedge E$ 9
12	<u>$a = b$</u>	
13	Ga	$=E$ 12, 10
14	\perp	$\perp I$ 13, 8
15	$\neg a = b$	$\neg I$ 12–14
16	<u>$Fc \vee Gc$</u>	
17	<u>Fc</u>	
18	$Fc \rightarrow a = c$	$\vee E$ 6
19	$a = c$	$\rightarrow E$ 18, 17
20	$a = c \vee b = c$	$\vee I$ 19
21	<u>Gc</u>	
22	$Gc \rightarrow b = c$	$\vee E$ 11
23	$b = c$	$\rightarrow E$ 22, 21
24	$a = c \vee b = c$	$\vee I$ 23
25	$a = c \vee b = c$	$\vee E$ 16, 17–20, 21–24
26	$(Fc \vee Gc) \rightarrow (a = c \vee b = c)$	$\rightarrow I$ 16–25
27	$\forall z((Fz \vee Gz) \rightarrow (a = z \vee b = z))$	$\forall I$ 26
28	$\neg a = b \wedge \forall z((Fz \vee Gz) \rightarrow (a = z \vee b = z))$	$\wedge I$ 15, 27
29	$\exists y[\neg a = y \wedge \forall z((Fz \vee Gz) \rightarrow (a = z \vee y = z))]$	$\exists I$ 28
30	$\exists x \exists y[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$	$\exists I$ 29
31	$\exists x \exists y[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$	$\exists E$ 2, 9–30
32	$\exists x \exists y[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))]$	$\exists E$ 1, 4–31

Derived rules

34

A. Offer proofs which justify the addition of the third and fourth CQ rules as derived rules.

Justification for the third rule:

1		$\neg\exists xAx$	
		<hr/>	
2			Aa
			<hr/>
3			$\exists xAx$ $\exists I$ 2
4			\perp $\perp I$ 3, 1
5		$\neg Aa$ $\neg I$ 2-4	
6		$\forall x\neg Ax$ $\forall I$ 5	

Justification for the fourth rule:

1		$\forall x\neg Ax$		
		<hr/>		
2			$\exists xAx$	
			<hr/>	
3				Aa
				<hr/>
4				$\neg Aa$ $\forall E$ 1
5				\perp $\perp I$ 3, 4
6			\perp $\exists E$ 2, 3-5	
7		$\neg\exists xAx$ $\neg I$ 2-6		