Logic Worksheet – Finale

Your name:	Mark (out of 5):
Logic class (A–F):	
Logic class tutor:	

This worksheet contains five questions, of roughly the sort you might expect to find in the formal component of the Part IA logic exam. The rubric for the exam is unchanged from previous years, and will be as follows: you have three hours to do three questions, and either one or two of these questions must come from the formal section of the paper. (The other question(s) will be more philosophical in nature.)

You should therefore treat this worksheet as providing you with mock exam questions, and you should try to do them in exam-like conditions. Do not try to do all of them in one sitting, but do:

- give yourself exactly one hour per question;
- work somewhere without distractions (e.g. a library);
- not look anything up while you are doing the questions.

Once you have completed them, self-mark all except those marked with a star (those will be marked by your logic class marker). The solutions will be available at http://www.nottub.com/forallx.shtml. Correct your own work *in red*, for the marker to review. In the box below, write something that you now firmly understand, as a result of doing these exercises:

Understand:

And in this box, write something that you want to know more about:

Want to know more about:

How you fill out these boxes will help to guide your logic class.

Each question should take an hour. A perfect answer would receive a notional 100 points. The number in square brackets after each component of a question designates the number of points that a full and correct answer to that component would merit. Questions marked with a star will be marked by your logic class marker.

1. Attempt all parts of this question.
   (a) Carefully define the following notions: [30]
      1. Truth-functional connectives
         A connective is a symbol which, combined with one or more (input) sentences, yields another sentence. A connective is truth-functional iff the truth value of the resulting sentence is determined (uniquely) by the truth values of the input sentences. For example: conjunction is a truth-functional connective, since a conjunction is true if and only if both conjuncts are true.
      2. Object language
         The object language, in a given context, is the language that is being discussed; it is the 'object' of study. In most of our work, the object language has been TFL or FOL.
      3. Metalanguage
         The metalanguage, in a given context, is the language that is used to talk about an object language. In most of our work, the metalanguage has been English, augmented with some convenient devices.
      4. Tautologically equivalent sentences
         Sentences $\mathcal{A}$ and $\mathcal{B}$ are are tautologically equivalent iff $\mathcal{A} \vDash \mathcal{B}$ and $\mathcal{B} \vDash \mathcal{A}$. Otherwise put, tautologically equivalent sentences have the same truth table.
      5. Provably equivalent sentences
         Sentences $\mathcal{A}$ and $\mathcal{B}$ are provably equivalent iff $\mathcal{A} \vdash \mathcal{B}$ and $\mathcal{B} \vdash \mathcal{A}$. Otherwise put, you can prove $\mathcal{B}$ from the assumption of $\mathcal{A}$, and *vice versa*.
   (b) Which of the following claims are true, and which are false? Explain your answers. [70]
      1. No valid argument has a premise which is necessarily true.
         False. If the conclusion of the argument is a tautology, then the argument will be valid whatever the premises are.
      2. No valid argument has a conclusion which is necessarily false.
         False. If one of the premises of the argument is a contradiction, then the argument will be valid whatever the conclusion is.
      3. Every valid argument with a false conclusion has at least one false premise.
         True. Valid arguments are truth-preserving: it is impossible for the premises of such all to be true, whilst the conclusion is false. So if the conclusion is false and the argument is valid, it must be that at least one of the premises is false.
      4. Every valid argument with a true conclusion has at least one true premise.

False. Consider the argument: All oranges are purple; all purple things are fruits; so all oranges are fruits.

5. If we add a new premise to a valid argument, the resulting argument is valid.
   True. If an argument is valid, it is impossible for all the premises to be true and the conclusion false. If we add a new premise, that cannot provide us with a way for all the premises—including the old ones—to be true and the conclusion false. (This property is known as *monotonocity*.)
6. If 'A, hence either B or C' is valid, then so is 'Neither C nor B, hence not-A'.
   True. For 'Neither C nor B, hence not-A' to be invalid, it would need to be possible for 'Neither C nor B' to be true whilst 'not-A' is false, i.e. for B and C to both be false whilst A is true. This valuation would make A true whilst making either B or C false, and hence make 'A, hence either B or C' invalid.
7. If 'A, hence if B then C' is valid, then so is 'both not-A and not-C, hence not-B'.
   False. Let both A and C be '$2 + 2 = 5$' and let B be '$2 + 2 = 4$'. Then the first argument is valid since its only premise is necessarily false, but the second argument is invalid since its only premise is necessarily true, whereas its conclusion is necessarily false.
8. Suppose we have a valid argument, with premises $\mathcal{A}_1, \ldots, \mathcal{A}_n$, and conclusion $\mathcal{C}$; then the argument with premises $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and conclusion $\mathcal{B} \vee \mathcal{C}$ is also valid (whatever $\mathcal{B}$ might be).
   True. If $\mathcal{B} \vee \mathcal{C}$ is false then $\mathcal{C}$ is false. So if it is impossible for $\mathcal{A}_1, \ldots, \mathcal{A}_n$ to be true whilst $\mathcal{C}$ is false, then it is also impossible for $\mathcal{A}_1, \ldots, \mathcal{A}_n$ to be true whilst $\mathcal{B} \vee \mathcal{C}$ is false.
9. If $\vDash \neg(\mathcal{A} \to \mathcal{B})$, then $\mathcal{A}$ is not contingent.
   True. The antecedent to this claim is that $\neg(\mathcal{A} \to \mathcal{B})$ is true on every valuation. A negated conditional is true iff the antecedent is true and the consequent is false. So $\mathcal{A}$ must be true on every valuation. But in that case, $\mathcal{A}$ must be a tautology; and no tautology is contingent.
10. If $\mathcal{A} \vdash \mathcal{B}$, then $\mathcal{B}$ is not contingent.
   False. Consider the case that $\mathcal{A}$ is '$P \wedge Q$' and $\mathcal{B}$ is 'Q'. Then certainly $\mathcal{A} \vdash \mathcal{B}$, just by using the rule $\wedge$E. But $\mathcal{B}$ may well be contingent in this case.

★ Any consistent set of sentences can be rendered inconsistent, by the addition of a new sentence.

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2. Attempt all parts of this question.
   (a) Formalise each of the following arguments in FOL as best you can. You must provide a symbolization key for each argument: [25]
      1. Everyone loves someone. So at least two people love each other.
         Symbolisation key:

domain: people
  Lxy: ______$_x$ loves ______$_y$
Symbolisation: $\forall x \exists y Lxy \therefore \exists x \exists y(\neg x = y \wedge Lxy \wedge Lyx)$

2. Everyone is an existentialist iff everyone is a poet. So Camus is an existentialist iff he is a poet.
   Symbolisation key:
   domain: people
     Ex: ______$_x$ is an existentialist
     Px: ______$_x$ is a poet
     c: Camus
   Symbolisation: $\forall x Ex \leftrightarrow \forall x Px \therefore Ec \leftrightarrow Pc$
3. The author of *L'Étranger* also wrote *Le mythe de Sisyphe*. So, at least one person wrote *Le mythe de Sisyphe*.
   Symbolisation key:
   domain: people
     Ex: ______$_x$ wrote *L'Étranger*
     Sx: ______$_x$ wrote *Le mythe de Sisyphe*
   Symbolisation: $\exists x((Ex \wedge \forall y(Ey \to x = y)) \wedge Sx) \therefore \exists x Sx$

★ Everybody loves my baby, but my baby don't love nobody but me. So I am my baby.

★ Every dog is a mammal. So every bark produced by a dog is a bark produced by a mammal.

(b) For each of the arguments you formalised in part (a) of this question, state whether it is valid or invalid. If it is valid, provide a natural deduction to show that it is valid. If it is invalid, provide a counterexample to show that it is invalid. [75]

1. Invalid. To produce a countermodel, suppose the domain contains exactly one woman, Anne, who loves herself. Then the premise is clearly true, and the conclusion is clearly false, since there is only one person in the domain.
2. Invalid. To produce a countermodel, suppose the domain contains two people: Camus and Kurt. Let Camus but not Kurt satisfy 'Px'; let neither Camus nor Kurt satisfy 'Ex'. Then '$\forall x Ex$' and '$\forall x Px$' both take the same truth value (false), so the premise is true. However, 'Ec' is true whilst 'Pc' is false, so the conclusion is false.
3. Valid. Here is a proof showing as much:

1	$\exists x((Ex \wedge \forall y(Ey \to x = y)) \wedge Sx)$	
2	$\quad (Ea \wedge \forall y(Ey \to a = y) \wedge Sa$	
3	$\quad Sa$	$\wedge$E 2
4	$\quad \exists x Sx$	$\exists$I 3
5	$\exists x Sx$	$\exists$E 1, 2–4

3. Attempt all parts of this question. You must use the proof system from the textbook.

(a) Show each of the following: [50]

1. $\vdash P \to (Q \to P)$

1	$\quad P$	
2	$\quad\quad Q$	
3	$\quad\quad P$	R 1
4	$\quad Q \to P$	$\to$I 2–3
5	$P \to (Q \to P)$	$\to$I 1–4

2. $\vdash (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

1	$\quad P \to (Q \to R)$	
2	$\quad\quad P \to Q$	
3	$\quad\quad\quad P$	
4	$\quad\quad\quad Q$	$\to$E 2, 3
5	$\quad\quad\quad Q \to R$	$\to$E 1, 3
6	$\quad\quad\quad R$	$\to$E 5, 4
7	$\quad\quad P \to R$	$\to$I 3–6
8	$\quad (P \to Q) \to (P \to R)$	$\to$I 2–7
9	$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$	$\to$I 1–8

★ $\vdash (\neg P \to \neg Q) \to (Q \to P)$

(b) Show both of the following: [50]

1. $\forall x Fx \leftrightarrow \exists x Gx \vdash \forall x Fx \vee \forall x \neg Gx$

1	$\forall x Fx \leftrightarrow \exists x Gx$	
2	$\quad \forall x Fx$	
3	$\quad \forall x Fx \vee \forall x \neg Gx$	$\vee$I 2
4	$\quad \neg \forall x Fx$	
5	$\quad\quad \exists x Gx$	
6	$\quad\quad \forall x Fx$	$\leftrightarrow$E 1, 5
7	$\quad\quad \bot$	$\bot$I 6, 4
8	$\quad \neg \exists x Gx$	$\neg$I 5–7
9	$\quad \forall x \neg Gx$	CQ 8
10	$\quad \forall x Fx \vee \forall x \neg Gx$	$\vee$I 9
11	$\forall x Fx \vee \forall x \neg Gx$	TND 2–3, 4–10

★ $\exists x Rax \rightarrow \exists x Rxa \vdash \exists x(Rab \rightarrow Rxa)$

4. Attempt all parts of this question.

   (a) Explain what it means to say that a relation is: [10]

   1. reflexive
      A relation, R, is reflexive iff $\forall x Rxx$
   2. symmetric
      A relation, R, is symmetric iff $\forall x \forall y(Rxy \rightarrow Ryx)$
   3. transitive
      A relation, R, is transitive iff $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$
   4. an equivalence relation
      A relation, R, is an equivalence relation iff it is reflexive, symmetric and transitive.

   (b) We say that a relation, R, is *negatively transitive* if and only if $\forall x \forall y \forall z((\neg Rxy \wedge \neg Ryz) \rightarrow \neg Rxz)$. Of the following relations, say which are reflexive, symmetric, transitive, and negatively transitive on the domain of people: [50]

   1. x is y's biological mother
      not reflexive; not symmetric; not transitive; not negatively transitive.
   2. x is y's only sibling
      not reflexive; symmetric; not transitive; not negatively transitive.
   3. x is no taller than y
      reflexive; not symmetric; transitive; negatively transitive
   4. x is a biological ancestor of y
      not reflexive; not symmetric; transitive; not negatively transitive.

   ★ x and y share no biological ancestor

   (c) Define the following terms: [10]

   1. subset
      A is a subset of B iff every member of A is a member of B, i.e.:
      $A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$
   2. proper subset
      A is a proper subset of B iff both A is a subset of B and some member of B is not in A, i.e.:
      $A \subset B \leftrightarrow (\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge \neg x \in A))$
   3. union
      The union of sets A and B is the set whose members are all and only those things which are either members of A or members of B, i.e.:
      $A \cup B = \{x \mid x \in A \vee x \in B\}$.
   4. intersection
      The intersection of sets A and B is the set whose members are all and only those things which are both members of A and members of B, i.e.:
      $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

5. Cartesian product
   The Cartesian product of two non-empty sets, A and B, is the set of all ordered pairs whose first component is any member of A and whose second component is any member of B, i.e. $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$.

(d) Attempt all parts of this question: [30]

1. Show that if A is a subset of B and B is a subset of C, then A is a subset of C.
   Suppose A is a subset of B and B is a subset of C. Consider any $x \in A$. Since A is a subset of B, $x \in B$. Since B is a subset of C, $x \in C$. So every member of A is a member of C, and hence A is a subset of C.
2. Does the statement in part d.1 of this question remain true, if we replace all instances of the word 'subset' with 'proper subset'? Explain your answer.
   It does. Suppose A is a proper subset of B and B is a proper subset of C. Then, since proper subsets are subsets, A is at least a subset of C, using exactly the same argument as before. To show that it is proper, observe the following: since B is a proper subset of C, there is something—e, let's say—which is a member of C but not a member of B; since A is a subset of B, e is not a member of A either. So A is indeed a proper subset of C.

★ Show that $A = B$ iff neither A nor B is a proper subset of $A \cup B$. Which axiom of set theory did you need to invoke to show this?

5. Attempt all parts of this question.

(a) Explain each of the following notions from probability theory: [10]

1. Conditional Probability
   The conditional probability of A given B, written $\Pr(A|B)$, is defined as follows:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

2. Independent Events
   Two events A and B (in an event space) are independent iff the probability of A is independent of the probability of B. More precisely, they are independent provided:

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

   It's worth noting that this is equivalent to both of the following, by the definition of conditional probability:

$$\Pr(A|B) = \Pr(A)$$
$$\Pr(B|A) = \Pr(B)$$

3. Exclusive Events
   Two events A and B (in an event space) are exclusive iff the events cannot both occur. More precisely, they are exclusive provided:

$$\Pr(A \cap B) = 0$$

4. Bayes's Theorem
   Bayes's Theorem is a result concerning conditional probability, as follows:

$$\Pr(A|B) = \frac{\Pr(A) \cdot \Pr(B|A)}{\Pr(B)}$$

(b) The inhabitants of Dance Island are staging their annual ballroom dance contest. For this, the inhabitants of the island are placed into dancing pairs, at random. [20]

1. You meet a dancing pair. The lead dancer of that pair was born in June. What is the probability that they were both born in June?
   In answering both parts of this question, I shall assume that birthdays are evenly distributed throughout all twelve months of the year, and that all birth events are independent from one another. Given both assumptions, the probability is simply $\frac{1}{12}$
2. You meet another dancing pair. One of them was born in June. What is the probability that they were both born in June?
   Let L be the event that Lead was born in June; let F be the event that follow was born in June. Then we simply apply Bayes's Theorem, using the same assumptions as before:

$$\begin{aligned}
\Pr(F \cap L | F \cup L) &= \frac{\Pr(F \cap L) \cdot \Pr(F \cup L | F \cap L)}{\Pr(F \cup L)} \\
&= \frac{\Pr(F \cap L)}{\Pr(F \cup L)} && \text{since } \Pr(F \cup L | F \cap L) = 1 \\
&= \frac{\Pr(F \cap L)}{\Pr(F \cap L) + \Pr(F \cap \overline{L}) + \Pr(\overline{F} \cap L)} && \text{expanding} \\
&= \frac{\frac{1}{12} \cdot \frac{1}{12}}{\frac{1}{12} \cdot \frac{1}{12} + \frac{1}{12} \cdot \frac{11}{12} + \frac{11}{12} \cdot \frac{1}{12}} && \text{given assumptions} \\
&= \frac{1}{1 + 11 + 11} = \frac{1}{23}
\end{aligned}$$

Flag any assumptions that you have had to make in answering the question. Also, comment on the relationship between your answers to (b)1 and (b)2.

I have already flagged the necessary assumptions. The probability for (b)2 is approximately half that of the probability for (b)1. Intuitively, this is because the setup in (b)2 provides us with much less specific information than the setup of (b)1. The setup of (b)2 rules out the possibility that neither dancer is born in June, but—unlike the setup in (b)1—does not set $\Pr(F \cap \overline{L}) = 0$.

(c) In the town of South Park, 20% of the population are children. 1% of the children have lice; among the rest of the population, 0.1% of them have lice. LNL (Lice or No Lice$^{\text{TM}}$) is a means for testing whether or not someone has a lice. If a person has lice, LNL will definitely say the person has lice. If a person does not have lice, there is a 5% chance that LNL will say (falsely) that the person does have lice (this is not affected by whether the test is performed on a child or on someone else). Calculate the probability that a randomly chosen inhabitant of South Park: [70]

In what follows, I shall use 'C' for the event that the person is a child, 'H' for the event that the person has lice, and 'T' for the event that LNL says that they have lice. I shall also frequently use the following rules:

$$\begin{aligned}\Pr(B) &= \Pr(A \cap B) + \Pr(\overline{A} \cap B) \\ &= \Pr(A) \cdot \Pr(B|A) + \Pr(\overline{A}) \cdot \Pr(B|\overline{A})\end{aligned}$$

which I shall refer to as 'expanding'.

1. has lice

$$\begin{aligned}\Pr(H) &= \Pr(C) \cdot \Pr(H|C) + \Pr(\overline{C}) \cdot \Pr(H|\overline{C}) && \text{expanding} \\ &= \frac{1}{5} \cdot \frac{1}{100} + \frac{4}{5} \cdot \frac{1}{1000} \\ &= \frac{14}{5000} = \frac{7}{2500}\end{aligned}$$

2. is a child, given that they have lice

$$\begin{aligned}\Pr(C|H) &= \frac{\Pr(C) \cdot \Pr(H|C)}{\Pr(H)} && \text{Bayes's Theorem} \\ &= \frac{\frac{1}{5} \cdot \frac{1}{100}}{\frac{7}{2500}} && \text{using previous answer} \\ &= \frac{5}{7}\end{aligned}$$

3. has lice, given that LNL says that they have lice

$$\begin{aligned}\Pr(H|T) &= \frac{\Pr(H) \cdot \Pr(T|H)}{\Pr(T)} && \text{Bayes's Theorem} \\ &= \frac{\Pr(H) \cdot \Pr(T|H)}{\Pr(H) \cdot \Pr(T|H) + \Pr(\overline{H}) \cdot \Pr(T|\overline{H})} && \text{expanding} \\ &= \frac{\frac{7}{2500} \cdot 1}{\frac{7}{2500} \cdot 1 + \frac{2500-7}{2500} \cdot \frac{1}{20}} \\ &= \frac{7 \cdot 20}{7 \cdot 20 + 2493} = \frac{140}{2633}\end{aligned}$$

★ has lice, given both that they are child and LNL says that they have lice