LECTURE 4

CAN A LAW OF LOGIC BE JUSTIFIED?

The rule of *modus tollens* can be derived from basic rules: $\to$E, $\bot$I and $\neg$I.
But then: how do we justify the *basic* rules?

4.1 Semantic justifications and rule-soundness

If we understand validity in terms of semantics, we will look for *rule-soundness*. Loosely, this is the property of 'never leading you from truths to falsehoods'. (See *Metatheory* §5.1 for the full definition.) Clearly rule-soundness is desirable.
Problem: to *prove* rule-soundness, we invoke the rules we want to justify! Or rather, we invoke *analogous* rules in the semantic metatheory. To illustrate:

Proof that $\wedge$*I is rule-sound.* Suppose we obtained line n by $\wedge$I. So the situation is:

i	$\mathcal{A}$	
j	$\mathcal{B}$	
n	$\mathcal{A} \wedge \mathcal{B}$	$\wedge$I i, j

Suppose also that, for all lines before line n, the undischarged assumptions for those lines semantically entail the sentences on those lines. Let v be any valuation that makes all the undischarged assumptions for line n true.

(a) The undischarged assumptions for line i are all among the undischarged assumptions for line n. By hypothesis, any valuation that makes all of the undischarged assumptions for line i true makes $\mathcal{A}$ true. v is such a valuation; so v makes $\mathcal{A}$ true.

(b) Exactly similar reasoning shows that v makes $\mathcal{B}$ true.

So ***from (a) and (b)***, v makes $\mathcal{A}$ true v makes $\mathcal{B}$ true. And so v makes $\mathcal{A} \wedge \mathcal{B}$ true. □

4.2 Inferentialist approaches

Logical inferentialists say that the meaning of the logical constants is given by their inferential roles. One possible motivation is the thought: *meaning is use*.

$$
\begin{array}{r|l|ll}
i & & \underline{\mathcal{A}} & \\
\vdots & & \vdots & \\
j & & \mathcal{B} & \\
\hline
j+1 & \multicolumn{2}{l}{\mathcal{A} \to \mathcal{B}} & \to\mathrm{I}\ i\text{–}j
\end{array}
\qquad\qquad
\begin{array}{r|ll}
i & \mathcal{A} & \\
j & \mathcal{A} \to \mathcal{B} & \\
n & \mathcal{B} & \to\mathrm{E}\ i, j
\end{array}
$$

NB: this implicitly invokes a notion of *core meaning*. For consider: 'once we have the concept of disjunction, our perceptions themselves may assume an irremediably disjunctive form' (Dummett 1991, p.267).

4.3 Two kinds of inferentialism

I-principles. Conditions when it is appropriate to *assert* a sentence.
Verificationist-ish idea: meanings are given by I-principles. 'the introduction rules for a constant **c** represent the direct or canonical means of establishing the truth of a sentence with principal operator **c**.' (Dummett 1991: 252)
Question: is a joke so funny that it's always *socially* appropriate a logical truth?

E-principle. Consequences which it is appropriate to *infer* from an asserted sentence.
Pragmatist-ish idea: the E-principles give the meanings. 'The underlying idea is that the content of a statement is what you can do with it if you accept it ... This is, of course, the guiding idea of a pragmatist meaning-theory.' (Dummett 1991: 280)
Question: if it's *socially* appropriate to infer God's existence from any claim, does that affect the meaning of 'and'?
Question: isn't pragmatism normally more *holist* than this?

4.4 Prior's Tonk

Prior's (1960) challenges inferentialism by defining a new connective, *Tonk*.

$$
\begin{array}{r|ll}
m & \mathcal{A} & \\
 & \mathcal{A} \pitchfork \mathcal{B} & \pitchfork\text{I } m
\end{array}
\qquad\qquad
\begin{array}{r|ll}
m & \mathcal{A} \pitchfork \mathcal{B} & \\
 & \mathcal{B} & \pitchfork\text{E } m
\end{array}
$$

Tonk is perfectly well-defined. But it is *trivialising*. It lets us infer anything from anything:

$$
\begin{array}{r|ll}
1 & \mathcal{A} & \\
\hline
2 & \mathcal{A} \pitchfork \mathcal{B} & \pitchfork\text{I } 1 \\
3 & \mathcal{B} & \pitchfork\text{E } 2
\end{array}
$$

So: we can't allow that *any* pair of I- and E-rules defines a logical operation.

4.5 Conservativeness

Belnap (1962) complains that Tonk is not **conservative**.
That is: we can use Tonk to derive statements which do *not* contain Tonk, which we couldn't have derived *without* using Tonk.
But classical negation is not conservative either! Consider

$$(A \rightarrow B) \lor (B \rightarrow A)$$

This sentence does not contain negation, and it cannot be proved in intuitionistic logic. But it is a theorem of classical logic (exercise: prove it using DNE).

But *actually*, I should say that classical negation is not conservative over $\{\lor, \rightarrow\}$.
That is: being Conservative is not *local*, but *relational*
And do we really have an *independent* motivation for insisting on Conservativeness?

4.6 Harmony and levelling of local peaks

Our inference rules come in pairs (intro/elim) and we want these to be *balanced*.
Idea of **harmony**: you can't get out more than you put in.
Sounds like a harmonious connective is one we should definitely be prepared to use.

Levelling of local peaks is a good stab at cashing out (some of) this idea.
(See Dummett *Logical Basis of Metaphysics* 1991: 247–8; Steinberger 2011.)
Let **c** be a logical constant.
A **local peak** for **c** is a use of **c**-introduction, immediately followed by **c**-elimination.
Local peaks can be **levelled** if we have a procedure for eliminating them.
Local peaks can be levelled for '∧', '∨' and '→'.

> *Proof that local peaks can be levelled for '∧'.* Suppose we have a local peak for ∧. So we have inferred $\mathcal{A} \wedge \mathcal{B}$ on line n by ∧I, and then inferred (e.g.) $\mathcal{A}$ on line $n + 1$ by ∧E. To get to line n, we must have inferred $\mathcal{A}$ on some earlier line i, and $\mathcal{B}$ on some earlier line j. But then there was no need for the detour through ∧I and ∧E: we can just look to line i instead of line $n + 1$. Graphically, do this (stars indicate need to shift some citations)

	Π_1	
i	$\mathcal{A}$	
	Π_2	
j	$\mathcal{B}$	
	Π_3	
n	$\mathcal{A} \wedge \mathcal{B}$	∧I i, j
$n + 1$	$\mathcal{A}$	∧E n
	Π_4	

	Π_1
i	$\mathcal{A}$
	Π_2
i	$\mathcal{B}$
	Π_3
	Π_4^*

We cannot level the local peak that makes tonk trivialising.
Can local peaks be levelled, for *classical* negation? Consider a local peak:

i	$\quad\mid \neg\mathcal{A}$	
‘	$\quad\mid \Pi$	
j	$\quad\mid \bot$	
$j+1$	$\neg\neg\mathcal{A}$	$\neg$I i–j
$j+2$	$\mathcal{A}$	DNE $j+1$

There is no guarantee that $\mathcal{A}$ has already been proved in some line prior to line i.
If we accept DNE, then local peaks for $\neg$ cannot be levelled.
So classical negation is not disharmonious!

But intuitionistic negation *is* harmonious:

	Π_1	
i	$\mathcal{A}$	
	Π_2	
j	$\quad\mid \mathcal{A}$	
	$\quad\mid \Pi_3$	
k	$\quad\mid \bot$	
$k+1$	$\neg\mathcal{A}$	$\neg$I j–k
$k+2$	$\bot$	$\bot$I i, $k+1$
	Π_4	

	Π_1
i	$\mathcal{A}$
	Π_2
	Π_3^*
$k-1$	$\bot$
	Π_4^*

So here is a *possible* argument for intuitionism: (a) argue for logical inferentialism, (b) argue that logical laws are acceptable iff they allow levelling of local peaks.
Still, there's *much* more to say…