LECTURE 2

TRUTH, PROOF, AND WARRANT

2.1 Constructivism in mathematics

Consider: *There are four cows in the field*; *There are four prime numbers between 0 and 10*.
Similar syntax, so similar semantics?
But if so, a philosophical question arise: what is the *nature* of numbers?
Platonists say they are *abstract* entities; but that is maybe odd.
Constructivists say they are in some sense *constructions*.
That sounds like a metaphysical claim. But it can have knock-on implications, e.g.:

Theorem. There are irrational numbers p and q such that p^q is rational.
Proof. $\sqrt{2}$ is irrational [proof elsewhere]. Now,
— If $\sqrt{2}^{\sqrt{2}}$ is rational, then let $p = q = \sqrt{2}$.
— If $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $p = \sqrt{2}^{\sqrt{2}}$ and $q = \sqrt{2}$ and note

$$p^q = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\times\sqrt{2})} = (\sqrt{2})^2 = 2$$

Either way, we have irrational p and q with p^q rational. □

The proof does not construct (specific) p and q! It's non-constructive.
And, formally, it relies on LEM/TND.

2.2 Mathematical truth as proof

Intuitionistic logic was initially introduced to underwrite constructivism. Roughly, say:

$$\text{it is true that } \mathcal{A} = \text{it is provable that } \mathcal{A}$$

Then we would get:

$$
\begin{aligned}
\mathcal{A} \wedge \mathcal{B} &\Rightarrow \text{it is provable that } \mathcal{A} \text{ and it is provable that } \mathcal{B} \\
\mathcal{A} \vee \mathcal{B} &\Rightarrow \text{either it is provable that } \mathcal{A} \text{ or it is provable that } \mathcal{B} \\
\mathcal{A} \to \mathcal{B} &\Rightarrow \text{there is a (constructive) method which} \\
&\qquad\qquad \text{turns any putative proof of } \mathcal{A} \text{ into a proof of } \mathcal{B} \\
\exists x \mathcal{A}(x) &\Rightarrow \text{for some specific } b \text{ it is provable that } \mathcal{A}(b) \\
\forall x \mathcal{A}(x) &\Rightarrow \text{there is a (constructive) method which} \\
&\qquad\qquad \text{acts on any given element } b \text{ and turns it into a proof that } \mathcal{A}(b) \\
\neg\mathcal{A} & \text{ is just defined as } \mathcal{A} \to \bot
\end{aligned}
$$

2.3 LEM and infinite domains

So now consider LEM:

$$
\begin{aligned}
\mathcal{A} \vee \neg\mathcal{A} \Rightarrow\ & \text{either it is provable that } \mathcal{A} \text{ or there is a (constructive) method which} \\
& \text{turns any putative proof of } \mathcal{A} \text{ into a proof of contradiction}
\end{aligned}
$$

Suppose we have an infinite domain, e.g. the natural numbers: $0, 1, 2, 3, \ldots$
Let $\mathcal{P}$ be a decidable predicate: of *each* number, we can decide whether $\mathcal{P}$ applies.
— But can we prove $\forall x \mathcal{P}(x)$??
— Equally, can we prove $\neg\forall x \mathcal{P}(x)$??
So LEM is not *guaranteed*. (But NB: not a *counterexample* to LEM!)
Helpful heuristic: classical semantics allows that $\forall x \mathcal{P}(x)$ can be true by *infinite coincidence*. BHK semantics doesn't allow that.

2.4 Focussing on the finite/infinite divide

If the domain had been finite, the above considerations would not have arisen.
In such cases, intuitionists allow '$\forall x \mathcal{P}(x) \vee \neg\forall x \mathcal{P}(x)$'.

So intuitionists worry that classical logic unacceptably transfers principles from finitely to infinitary cases.

But what's the principle being invoked here? The finite domain might be *unfeasibly large*. The thought is counterfactual (prov*able*). It's something like this: *if we inspected each object in the domain in turn, we could determine whether $\mathcal{P}$ applied or not; and so 'in the end' we would have checked them all; and that would be sufficient either to license $\forall x \mathcal{P}(x)$ or $\neg \forall x \mathcal{P}(x)$.* The form of this argument is this:

$$\mathcal{T} \mathrel{\Box\!\!\rightarrow} (\mathcal{A} \vee \neg \mathcal{A}) \therefore (\mathcal{T} \mathrel{\Box\!\!\rightarrow} \mathcal{A}) \vee (\mathcal{T} \mathrel{\Box\!\!\rightarrow} \neg \mathcal{A})$$

This is invalid! If I were to toss this fair coin, either it would land heads or tails; but …
Dummett (1973: 245) notes this, but suggests that the principle is ok in *mathematics* as:

- we don't need to *supplement* the antecedent with more information
- there is no risk of *indeterminacy*
- carrying out the test will not change the result

2.5 But why identify truth with (constructive) proof?

Motivation 1: I just want to keep track of *constructive* reasoning!
Ok, but distinguish two senses of the word 'logic'. (i) An (essentially) arbitrarily defined relation between sentences, to be studied by mathematicians. (ii) The thing that underpins *reasoning*.

Motivation 2: we think that mathematical objects are constructed by mathematicians.
O RLY? 'It looks at first glance, though, in these cases, we have a metaphysical doctrine yielding consequences for logic; the difficulty is in seeing how one could decide for or against the metaphysical premiss. We also face another and greater difficulty: to comprehend the content of the metaphysical doctrine. What does it mean to say that natural numbers are mental constructions, or that they are independently existing immutable and immaterial objects? What does it mean to ask whether or not past or future events are *there*? What does it mean to say, or deny, that material objects are logical constructions

out of sense-data? In each case, we are presented with alternative *pictures*. The need to choose between these pictures seems very compelling; but the non-pictorial content of the pictures is unclear.' (Dummett 1991: 10)

Motivation 3: mathematical *truth* looks spooky, *unless* identified with proof.
Dummett develops this, focussing on **Manifestation** and **Acquisition**.

2.6 Empirical truth as warranted assertibility

What's the *empirical* analogue of a mathematical proof?
Maybe *warrant*, but warrant is insufficient for truth. Künne's example: the number of hairs on my head is either even or odd …
Just as we considered provability, we need to consider warrantability. Roughly:

$$\text{it is true that } \mathcal{A} = \text{it is warrantable that } \mathcal{A}$$

We would now clauses just like the BHK interpretation. And now, concerning LEM:

- A city will never be built on this spot
- There are no intelligent extraterrestrials
- Julius Caesar shaved himself the morning he crossed the Rubicon

'We are entitled to say that a statement P must be either true or false … only when P is a statement of such a kind that we could in a finite time bring ourselves into a position in which we were justified either in asserting or in denying P; that is, when P is an effectively decidable statement.' And 'What I have done here is to transfer to ordinary statements what the intuitionists say about mathematical statement.' (1957: 160)
But it is worth wondering which of the three earlier motivations transfer across.

2.7 Difficulties with the analogy

There seems to be no epistemic standard, for empirical sciences, which plays the same role as provability in mathematics.

Problem 1. *Provability* is factive; is *warrantibility*?
Serious threat: it might be a general principle that *any* empirical demonstration could, in principle, be overturned in the light of future evidence.
Could retreat, I suppose, to immediate reports of sensation. But that sounds terrible.

Problem 2. The principled distinction between decidable and undecidable cases – motivated neatly in the mathematical case by considering atomic claims and then quantification – is ill-suited to empirical cases.

> there is no fundamental difference between a universal sentence and a particular sentence with regard to verifiability but only a difference in degree. Take for instance the following sentence: "There is a white sheet of paper on this table". In order to ascertain whether this thing is paper, we may make a set of simple observations and then, if there still remains some doubt, we may make some physical and chemical experiments. Here as well in the case of the law, we try to examine sentences which we infer from the sentence in question. These inferred sentences are predictions about future observations. The number of such predictions which we can derive from the sentence given is infinite; and therefore the sentence can never be completely verified. (Carnap 1936: 425)

If warrantability were *knowability*, some issues might be resolved. But:

- Are there good reasons to identify truth with warrantability?
- How would we deal with the problems posed by counterfactuals?

More on the latter next time.