LECTURE 1

NATURAL DEDUCTION

Course aim: to challenge the hegemony of classical logic.
Specific focus: natural deduction and intuitionistic logic.

1.1 Semantic approaches

We can characterise classical logic *semantically*. We have:
Principle of Bivalence. *Every sentence is either true or false (always one, never both)*
We define characteristic truth tables, and use them to define (classical) entailment:

> $\Gamma \vDash_C C$ **iff** there is no valuation which, when evaluated according to the classical truth tables, makes all of Γ true whilst making C false.

So we can produce non-classical logics by abandoning the Principle of Bivalence.
Example 1: sentences can be indeterminate. Then we might use (Strong) Kleene's truth-tables (reading 'I' as 'indeterminate'):

$\neg$	
T	F
I	I
F	T

$\vee$	T	I	F
T	T	T	T
I	T	I	I
F	T	I	F

$\wedge$	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

$\rightarrow$	T	I	F
T	T	I	F
I	T	I	I
F	T	T	T

When $\mathcal{A} \Rightarrow$ I, we have $(\mathcal{A} \vee \neg\mathcal{A}) \Rightarrow$ I; so LEM is *not a theorem* of Strong Kleene Logic.
Example 2: sentences can be both true and false. With Priest, we might well offer the *same* three-valued truth tables, but reading 'I' as 'both' (so that LEM *is* a theorem).
Example 3: sentences can be true, or false, or both, or neither. Then we would look to a four-valued logic such as FDE.
I could multiply examples, *but I won't be doing any of this!*

1.2 Deductive approaches

We can also characterise classical logic *deductively*.
We specify some basic, classically acceptable, natural deduction rules: $\wedge$I, $\wedge$E, $\vee$I, $\vee$E, $\rightarrow$I, $\rightarrow$E, $\leftrightarrow$I, $\leftrightarrow$E, $\bot$I, $\bot$E, $\neg$I, TND. Then use them to define (classical) provability:

> $\Gamma \vdash_{\text{C}} \mathcal{C}$ **iff** there is some natural deduction which starts with assumptions among Γ, uses only the classically acceptable rules, and then ends with $\mathcal{C}$ (on no further undischarged assumptions).

By **Soundness** and **Completeness**, the semantic and deductive characterisations 'come to the same'. (For any sentences Γ and any sentence $\mathcal{C}$, we have: $\Gamma \vDash_{\text{C}} \mathcal{C}$ iff $\Gamma \vdash_{\text{C}} \mathcal{C}$.)
But we can produce non-classical logics by laying down different deduction rules.

1.3 Intuitionistic logic, defined

Intuitionist logic accepts all of the basic classical rules *except* TND. We say:

> $\Gamma \vdash_{\text{I}} \mathcal{C}$ **iff** there is some natural deduction which starts with assumptions among Γ, uses only the intuitionistically acceptable rules, and then ends with $\mathcal{C}$ (on no further undischarged assumptions).

The next three lectures will discuss the *philosophy* behind the logic.
NB: some important upshots:

$$
\begin{aligned}
\text{we lose LEM}: &\quad \nvdash_{\text{I}} (\mathcal{A} \vee \neg\mathcal{A}) \\
\text{we lose DNE}: &\quad \neg\neg\mathcal{A} \nvdash_{\text{I}} \mathcal{A} \\
\text{we lose Peirce's Law}: &\quad \nvdash_{\text{I}} ((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{A}) \rightarrow \mathcal{A} \\
\text{we lose a DeM rule}: &\quad \neg(\mathcal{A} \wedge \mathcal{B}) \nvdash_{\text{I}} \neg\mathcal{A} \vee \neg\mathcal{B} \\
\text{we lose a CQ rule}: &\quad \neg\forall x\varphi(x) \nvdash_{\text{I}} \exists x\neg\varphi(x)
\end{aligned}
$$

But NB: $\neg(\mathcal{A} \vee \neg\mathcal{A})$ *is an intuitionistic contradiction*. So intuitionists don't accept that LEM is a law of logic. But they *don't* (and can't) claim to have a counterexample to LEM!

1.4 Neat properties of intuitionistic logic

Disjunction Property (Gödel 1932). *If* $\vdash_I (\mathcal{A} \vee \mathcal{B})$*, then either* $\vdash_I \mathcal{A}$ *or* $\vdash_I \mathcal{B}$.
This is clearly false for classical logic: take any instance of LEM.
This makes intuitionistic logic more 'constructive' than classical logic.

Not finitely-valued (Gödel 1933). *Intuitionistic propositional logic cannot be interpreted using finitely many different truth values.*
Proof sketch. Suppose it is three-valued. Then any interpretation will make (at least) two among these atomic sentences – 'A', 'B', 'C' or 'D' – have the same truth value. So this would be a tautology:

$$(A \leftrightarrow B) \vee (A \leftrightarrow C) \vee (A \leftrightarrow D) \vee (B \leftrightarrow C) \vee (B \leftrightarrow D) \vee (C \leftrightarrow D)$$

By the Disjunction Property, intuitionistic logic proves one of the six biconditionals; but it clearly doesn't! And the proof generalises easily for numbers greater than three. □

Equiconsistency (Gödel 1933). *Intuitionistic predicate logic and classical predicate logic are equiconsistent.*
Proof sketch. Define a translation as follows:

$$\begin{aligned}
\mathcal{A}^{\mathrm{g}} &= \neg\neg\mathcal{A}, \text{ if } \mathcal{A} \text{ is atomic} \\
(\mathcal{A} \wedge \mathcal{B})^{\mathrm{g}} &= (\mathcal{A}^{\mathrm{g}} \wedge \mathcal{B}^{\mathrm{g}}) \\
(\mathcal{A} \vee \mathcal{B})^{\mathrm{g}} &= \neg(\neg\mathcal{A}^{\mathrm{g}} \wedge \neg\mathcal{B}^{\mathrm{g}}) \\
(\mathcal{A} \to \mathcal{B})^{\mathrm{g}} &= (\mathcal{A}^{\mathrm{g}} \to \mathcal{B}^{\mathrm{g}}) \\
(\neg\mathcal{A})^{\mathrm{g}} &= \neg\mathcal{A}^{\mathrm{g}} \\
(\forall x\mathcal{A})^{\mathrm{g}} &= \forall x\mathcal{A}^{\mathrm{g}} \\
(\exists x\mathcal{A})^{\mathrm{g}} &= \neg\forall x\neg\mathcal{A}^{\mathrm{g}}
\end{aligned}$$

We then prove that, if $\vdash_C \mathcal{A}$, then $\vdash_I \mathcal{A}^{\mathrm{g}}$. This is by showing that each classical rule is mirrored by a 'translated' derived rule in intuitionistic logic. □