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On an analytical solution for the damped Hertzian spring

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Abstract – Contact models including an elastic and a dissipative force are widely used to model inelastic granular collisions. Whilst there exists an analytical solution for the linear spring-dashpot (LSD) model, for the more practically important non-linear models often only the asymptotic behaviour is known. In this letter, we demonstrate that if the Hertzian contact model is complemented by a damping force proportional to 1/4-power of the normal deformation, it is possible to map its behaviour onto the LSD model and hence obtain simple analytical relationships between the model parameters and the characteristics of the granular collision. In practice, this allows the prediction of parameters for discrete element models of powder flow and compaction by using experimental observables such as restitution coefficient and collision time as input data.

Introduction. – Modelling granular flow in applications ranging from processing pharmaceutical powders [1] to industrial mills and fluidised beds requires realistic description of binary collisions between the particles [2]. Even if their shape is assumed to be spherical, the choice of the constituent force laws and their parametrisation is not a simple task [3–6]. A number of observables such as collision time, restitution coefficient, maximum and residual deformations can be used to get insight into the interplay between elastic, plastic, viscous and other physical phenomena relevant to the collision process. As it is often difficult to choose between various models [4,7] when they all can adequately replicate the experimental observables within the region where the model parameters were fitted, choosing the most computationally simple model might be advantageous in this case.

In this letter, we concentrate on two relatively simple and widely used phenomenological models: 1) a linear elasticity model with a linear damping force, also known as the linear spring-dashpot (LSD) model, and 2) a non-linear damped Hertzian spring-dashpot (HSD) model, describing an inelastic collision of two spheres. Both models predict a restitution coefficient independent of the impact velocity but differ in their prediction of the force-deformation relationship and the collision time. Whilst the linear model predicts collision time to be independent of the impact velocity, its non-linear counterpart predicts shorter collision times at higher impact velocities, which agrees with the experiments on particle collision. The difficulty with using the more realistic [8] non-linear model is the lack of analytical relationships between the model input parameters such as damping coefficient and the experimental observables such as the coefficient of restitution, which is the issue we would like to resolve in this letter. Despite the absence of a simple analytical solution for the HSD model, we demonstrate that a number of useful analytical relationships can still be derived.

The equation of motion describing the collision of two bodies of masses m_1 and m_2 can be written in the most general form as

$$m\ddot{\delta} + F_{\text{dis}}(\delta, \dot{\delta}) + F_{\text{cons}}(\delta) = 0, \tag{1}$$

where $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass, $F_{\text{dis}}(\delta, \dot{\delta})$ is the total dissipative force and $F_{\text{cons}}(\delta)$ is the total conservative force including, but not limited to elasticity. Here positive definite deformation δ is a function of the particle radii r_1 and r_2 and the separation of their centres d defined as

$$\delta = \max[0, r_1 + r_2 - d]. \tag{2}$$

One of the simplest contact models in common use is the linear spring-dashpot (LSD) model for which Hooke's law

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is used to describe $F_{\mathrm{cons}}(\delta) = k\delta$ and eq. (1) becomes

$$\ddot{\delta} + 2\gamma\dot{\delta} + w_0^2\delta = 0, \tag{3}$$

where $w_0 = \sqrt{k/m}$ is the frequency of the undamped harmonic oscillator and the damping coefficient γ is responsible for energy dissipation. The LSD model is widely used due to the existence of simple analytical relationships between parameters k and γ and experimental observables such as collision time τ and restitution coefficient ε, with the latter defined as the ratio of the relative velocities after and before the collision. In the notation of eq. (1), the restitution coefficient is defined as

$$\varepsilon = -\frac{\dot{\delta}(\tau)}{\dot{\delta}(0)}. \tag{4}$$

For the under-damped case in which $w_0 > \gamma$, the initial conditions $\delta(0) = 0$ and $\dot{\delta}(0) = v_0$ result in the following solutions for the deformation:

$$\delta(t) = \frac{v_0}{w_1}\exp(-\gamma t)\sin(w_1 t) \tag{5}$$

and deformation rate:

$$\dot{\delta}(t) = \frac{v_0}{w_1}\exp(-\gamma t)\left[w_1\cos(w_1 t) - \gamma\sin(w_1 t)\right], \tag{6}$$

where $w_1 = \sqrt{w_0^2 - \gamma^2}$ is the frequency of the damped oscillator. If the duration of the collision is taken as half the period of the oscillation, defined as $\delta(\tau) = 0$ in eq. (5), then the collision time is $\tau = \pi/w_1$. Using this value for τ in eq. (6) and substituting it into eq. (4) gives

$$\varepsilon = \exp\left(-\frac{\gamma\pi}{w_1}\right). \tag{7}$$

The inverse relationship

$$\gamma = \frac{-\ln\varepsilon}{\sqrt{\ln^2\varepsilon + \pi^2}}w_0 \tag{8}$$

allows us to choose the correct value for parameter γ in eq. (3) if the experimental value for ε is known. Since the impact velocity v_0 does not enter eq. (8), the LSD model predicts that the restitution coefficient is independent of v_0. The same is true for the collision time $\tau = \pi/\sqrt{w_0^2 - \gamma^2}$, which depends only on the parameters of eq. (3) but not on v_0.

In a collision of two elastic spheres with radii r_1 and r_2, the integration of Hooke's law over the area of deformation results in a nonlinear relationship known as Hertz's law:

$$F_{\mathrm{Hertz}}(\delta) = \frac{4E_{\mathrm{eff}}\sqrt{r_{\mathrm{eff}}}}{3}\delta^{\frac{3}{2}}, \tag{9}$$

where $1/E_{\mathrm{eff}} = (1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2$ is a function of elastic properties of the bulk materials with Young's moduli E_1 and E_2 and Poisson's ratios ν_1 and ν_2 and $r_{\mathrm{eff}} = r_1 r_2/(r_1 + r_2)$ is the effective radius. Equation (9) is true for relatively small deformations and it assumes that material properties are isotropic and do not change under load. It can also be extended to include, for example, cohesive interactions [9,10] not included in our simple model.

If eq. (9) is used to describe the conservative force in eq. (1), the expression for the dissipative force must also be non-linear to ensure that the restitution coefficient ε does not increase with the impact velocity [11]. Experimental data often indicates that ε decreases with the impact velocity [5], albeit very slowly. Therefore, one of the two most commonly used non-linear expressions for the dissipative force provides a decreasing restitution coefficient as described, for example, in refs. [12] and [13], whereas the other results in a constant restitution coefficient as described below.

In 1992, Tsuji *et al.* [14] demonstrated that a dissipative force of form

$$F_{\mathrm{dis}}(\delta) = \alpha(\varepsilon)\sqrt{mK}\delta^{\frac{1}{4}}\dot{\delta}, \tag{10}$$

where $K = 4/3E_{\mathrm{eff}}\sqrt{r_{\mathrm{eff}}}$ is the coefficient in front of the $\delta^{\frac{3}{2}}$ term in eq. (9), similar to the LSD model above, will result in a velocity-independent coefficient of restitution. As there were no analytical expression for $\alpha(\varepsilon)$ found at the time, the suggested recipe for choosing parameter α was simply to read its value of $\alpha(\varepsilon)$ graph calculated numerically and shown in fig. 4 in the original manuscript or to use a value from table 1 in ref. [15]. This approach or direct numerical integration was widely adopted since for adjusting the damping coefficient to obtain the desired coefficient of restitution.

In this letter we demonstrate how to map the Hertzian spring-dashpot (HSD) model described by eqs. (9) and (10) onto the LSD model and hence find the analytical solution for $\alpha(\varepsilon)$.

Solution for Hertzian spring-dashpot (HSD) model. – Similarly to eq. (3), the equation of motion for the damped Hertzian oscillator can be written as

$$\ddot{\delta} + \alpha(\varepsilon)\Omega_0\delta^{1/4}\dot{\delta} + \Omega_0^2\delta^{3/2} = 0, \tag{11}$$

where parameter $\Omega_0 = \sqrt{K/m}$ has dimensions of $1/(\mathrm{sm}^{1/4})$ and parameter $\alpha(\varepsilon)$ is identical to that in the original work by Tsuji *et al.* [14]. Since $v(t) = \frac{\mathrm{d}\delta}{\mathrm{d}t}$ and $\ddot{\delta} = \frac{\mathrm{d}\delta}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}\delta} = v\frac{\mathrm{d}v}{\mathrm{d}\delta}$, eq. (11) can also be written as a set of coupled differential equations:

$$v\frac{\mathrm{d}v}{\mathrm{d}\delta} + \alpha(\varepsilon)\Omega_0\delta^{1/4}v + \Omega_0^2\delta^{3/2} = 0, \tag{12}$$

$$v(t) = \frac{\mathrm{d}\delta}{\mathrm{d}t}. \tag{13}$$

Consider substituting the function $\delta(t)$ by a function $x(t)$ such as

$$\delta = Ax^n \quad \text{and} \quad \mathrm{d}\delta = nAx^{n-1}\mathrm{d}x. \tag{14}$$

This transforms eq. (12) into

$$v\frac{\mathrm{d}vA^{-1}x^{1-n}}{n\mathrm{d}x}+\alpha(\varepsilon)\Omega_0A^{1/4}x^{n/4}v+\Omega_0^2A^{3/2}x^{3n/2}=0.$$

Note that if $1-n=n/4$, which is true for $n=4/5$, the powers of x become the same for the first and the second terms, and division of all three terms by $x^{1/5}$ gives

$$v\frac{5\mathrm{d}vA^{-1}}{4\mathrm{d}x}+\alpha(\varepsilon)\Omega_0A^{1/4}v+\Omega_0^2A^{3/2}x=0. \tag{15}$$

The next step is to choose A such that the coefficients in front of the first and the third terms become identical, *i.e.* $5A^{-1}/4=A^{3/2}$ or $A=(5/4)^{2/5}$, which after substitution and normalisation gives simply

$$v\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{2}{\sqrt{5}}\alpha(\varepsilon)\Omega_0v+\Omega_0^2x=0. \tag{16}$$

This equation can be directly compared to eq. (3) for the LSD model if, similarly to eq. (8), one uses

$$\alpha(\varepsilon)=\frac{-\sqrt{5}\ln\varepsilon}{\sqrt{\ln^2\varepsilon+\pi^2}}. \tag{17}$$

This is the function depicted as $\varepsilon(\alpha)$ in fig. 4 of the original work by Tsuji *et al.* [14].

Note that time t does not enter eq. (16) explicitly and it can be viewed as a differential equation for the phase space trajectory $v(x)$. Therefore, if numerical values of w_0 and Ω_0 are identical, the phase space trajectory for the Hertzian oscillator $v(\delta)$ can be obtained from that for the corresponding LSD model by rescaling the coordinate axis while keeping the velocity axis unchanged.

Mapping the HSD model onto the LSD model. – Assuming that all the parameters of the HSD model are known, we shall now demonstrate how to calculate its phase space trajectory $v(\delta_{\mathrm{HSD}})$ analytically. To ensure that eqs. (16) and (3) are identical, we assume that the mass m and the impact velocity v_0 in the corresponding LSD model are identical to those in the HSD model. A further necessary condition is the equality of the numerical values for w_0 and Ω_0 (or $\sqrt{k/m}$ and $\sqrt{K/m}$) which is automatically satisfied if the unit of length is set to $(\frac{k}{K})^2$. For a collision between two identical spheres of diameter d, $r_{\mathrm{eff}}=d/4$ and $K=2/3E_{\mathrm{eff}}\sqrt{d}$, it is convenient to set the unit length as d, as it is independent of the impact velocity, by adjusting the spring constant k such that $(\frac{k}{K})^2=d$. The latter condition implies that $k=2/3E_{\mathrm{eff}}d$ with which eqs. (16) and (3), when expressed in reduced units (length being measured in units of d and velocity in units of v_0), become identical to each other with

$$w_0^*=\Omega_0^*=\frac{2}{\sqrt{\pi}}\frac{\sqrt{E_{\mathrm{eff}}/\rho}}{v_0}, \tag{18}$$

where ρ is the particle density.

For practical applications, the complete procedure for finding $v(\delta_{\mathrm{HSD}})$ is as follows. First, the solution to the LSD model with $k=2/3E_{\mathrm{eff}}d$, restitution coefficient ε, mass m and impact velocity v_0 identical to those of the HSD model should be found. Second, the deformation δ_{LSD} should be rescaled such that

$$\frac{\delta_{\mathrm{HSD}}}{d}=(5/4)^{2/5}\left(\frac{\delta_{\mathrm{LSD}}}{d}\right)^{4/5}, \tag{19}$$

while the corresponding velocities remain unchanged.

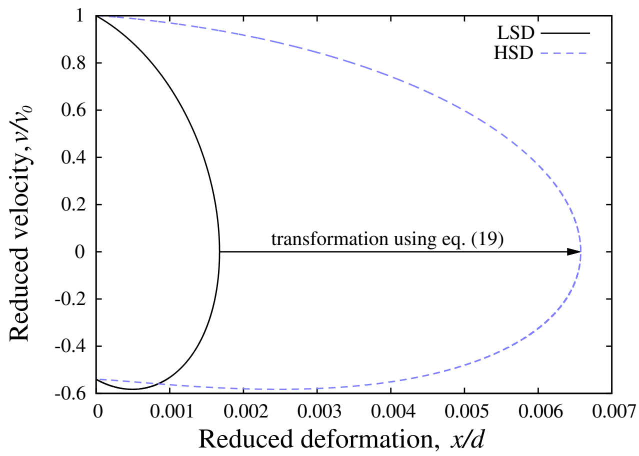

Fig. 1: (Colour on-line) The deformation-velocity phase space trajectory during a collision of two particles described by the LSD (solid line) and HSD (dashed line) and both characterised by a restitution coefficient of 0.54.

As an example, we demonstrate the deformation-velocity phase plane calculated for a collision of two 1 cm spheres made of compacted microcrystalline cellulose (MCC), a common pharmaceutical excipient used to produce tablets. The restitution coefficient of 0.54 was deduced from a series of ball-drop experiments, and Young's modulus $E=6\,\mathrm{GPa}$ [16] and Poisson's ratio $\nu=0.3$ were used to produce data shown in fig. 1. The solid line in fig. 1 corresponds to the equivalent LSD model calculated as described above, whereas the dashed line was obtained using eq. (19). Both collisions start at the upper left corner where the deformation is zero and the velocity is at its maximum value. The maximum deformation is then reached as velocity goes through zero, after which the rebound process starts as indicated by the negative values of velocity.

Maximum deformation. – Using energy conservation, it is straightforward to calculate the maximum deformation in a collision described by either the LSD or HSD model without dissipation. However, if some dissipation is present, then the work done by the dissipative forces is difficult to integrate. Since the maximum deformation is achieved when the velocity turns zero, for the LSD model eq. (6) predicts that this will take place at time

$$t=\frac{1}{w_1}\arctan\frac{w_1}{\gamma}. \tag{20}$$

By substituting this time into eq. (5) and taking into account that $w_0^2 = w_1^2 + \gamma^2$, we obtain the maximum deformation for the LSD model:

$$\delta_{\max}^{\mathrm{LSD}} = \frac{v_0}{w_0} \exp\left(-\frac{\gamma}{w_1} \arctan \frac{w_1}{\gamma}\right). \tag{21}$$

Using eq. (4), for a nearly elastic collision when $\gamma \ll w_0$, eq. (21) can be approximated by simply

$$\delta_{\max}^{\mathrm{LSD}} \approx \frac{v_0}{w_0}\sqrt{\varepsilon}. \tag{22}$$

The maximum deformation for the HSD model can be calculated by applying the transformation (eq. (19)) derived in the previous section.

Collision time. – Alongside with the coefficient of restitution, the time of the collision, τ, is often used for model parametrisation of granular collisions [4]. As we demonstrated in the introduction, τ does not depend on the impact velocity v_0 for the LSD model and is given as $\tau = \pi/\sqrt{w_0^2 - \gamma^2}$. The collision time of two elastic spheres without any dissipation can be calculated analytically [17] as

$$\tau_{\mathrm{Hertz}} = 2.214 \left(\frac{\rho}{E_{\mathrm{eff}}}\right)^{2/5} \frac{d}{v_0^{1/5}}, \tag{23}$$

which decreases as v_0 is increased.

It is possible to calculate the collision time from the phase trajectory, as $\tau = \sum \Delta\tau_i = \sum \frac{\Delta x_i}{v_i}$, where the sum is taken along the phase trajectory on both compression and rebound paths. At a given ε, the LSD model predicts that both the deformation and its rate are linearly proportional to v_0 (see eqs. (5) and (6)). Therefore, as the transformation of the space trajectory is affine, the ratios $\frac{\Delta x_i}{v_i}$ are not affected by varying v_0 and the collision time remains constant. For the HSD model, however, the x axis changes in proportion to $v_0^{4/5}$ according to eqs. (5) and (19), while the v axis still varies in proportion to v_0:

$$\tau_{\mathrm{HSD}} \propto \sum \frac{\Delta\left(\frac{v_0}{w_1}\exp(-\gamma t_i)\sin(w_1 t_i)\right)^{4/5}}{v_i}, \tag{24}$$

where t_i is the time measured for the LSD model when its phase coordinates are $(x_i; v_i)$. The net effect for the inelastic collision with arbitrary ε is that $\tau \sim 1/v_0^{1/5}$, which is consistent with eq. (23) for the special case of $\varepsilon = 1$. This means that, similarly to eq. (23), for any given ε the collision time as a function of v_0 is a straight line with a slope of $-1/5$ when plotted on a double logarithmic scale. Figure 2 shows this for the case of $\varepsilon = 0.1$ and $\varepsilon = 0.01$ together with the completely elastic collision for which $\varepsilon = 1$. Here we have kept all HSD model parameters for the MCC granule unchanged except for the restitution coefficient and calculated the collision times numerically by integrating the equation of motion. The set of horizontal lines in fig. 2 shows collision times for the corresponding LSD models.

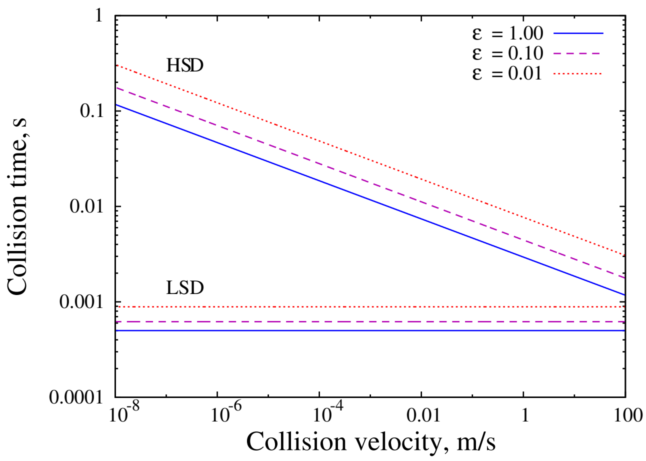

Fig. 2: (Colour on-line) Collision times calculated using the LSD (horizontal lines) and HSD (inclined lines) models for different values of the restitution coefficient ε.

Both linear and non-linear models predict that higher dissipation or lower values of restitution coefficient ε increases the collision time as demonstrated by an upward shift of the lines in fig. 2. For the LSD model, the ratio between the collision time for an oscillator described by restitution coefficient ε, $\tau_{\mathrm{LSD}}(\varepsilon)$, to that for a completely elastic collision, $\tau_{\mathrm{LSD}}(1)$, is equal to w_0/w_1 or in terms of ε:

$$\frac{\tau_{\mathrm{LSD}}(\varepsilon)}{\tau_{\mathrm{LSD}}(1)} = \frac{\sqrt{\ln^2\varepsilon + \pi^2}}{\pi}. \tag{25}$$

Despite the analytical expression for $\delta^{\mathrm{HSD}}(t)$ not being available even for undamped Hertzian spring, it is relatively simple to find the asymptotic behaviour of the collision time for the lightly damped HSD model, *i.e.* at ε close to 1. By rewriting eq. (24) as

$$\tau_{\mathrm{HSD}} \propto \sum \frac{\Delta\delta_i^{\mathrm{LSD}}}{v_i}\left(\frac{v_0}{w_1}\exp(-\gamma t_i)\sin(w_1 t_i)\right)^{-1/5}, \tag{26}$$

and noting that $w_1 = w_0$ and $\exp(-\gamma t_i)^{-1/5} = 1 + \gamma t_i/5$ at first-order approximation, we obtain

$$\tau_{\mathrm{HSD}} \propto \sum \frac{\Delta\delta_i^{\mathrm{LSD}}}{v_i}\left(\frac{v_0}{w_0}\sin(w_0 t_i)\right)^{-1/5}\left(1 + \frac{\gamma t_i}{5}\right). \tag{27}$$

The multipliers $(1 + \gamma t_i/5)$ describe the collision time increase along the phase trajectory compared to the completely elastic case when $\gamma = 0$. Since the phase trajectory of an elastic collision is the same on the compression and the rebound path, there will be two identical fractions $\frac{\Delta x_i}{v_i}$ entering the sum shown in eq. (27), one multiplied by $(1 + \gamma t_i/5)$ and the other multiplied by $(1 + \gamma(\tau_{\mathrm{LSD}}(0) - t_i)/5)$ (here we used the fact that collision time does not change with ε at first-order approximation according to eq. (25)). By rearranging the terms in

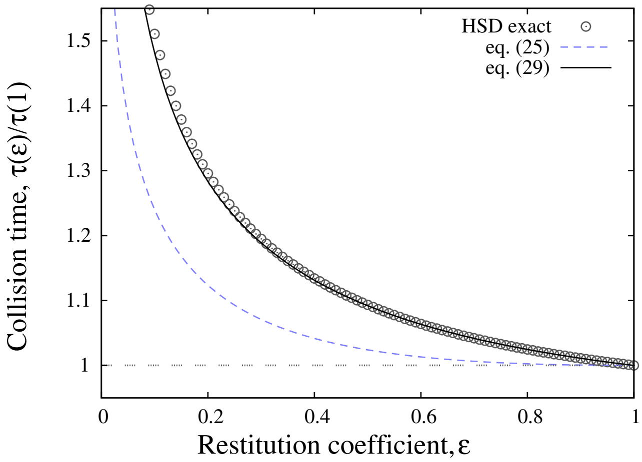

Fig. 3: (Colour on-line) Collision time increase due to the dissipative forces in the LSD (the dashed line) and HSD (circles and the solid line) models.

eq. (27), grouping them together and taking into account eq. (4) and that $\tau_{\mathrm{LSD}}(0) = \pi/w_0$ we obtain

$$\frac{\tau_{\mathrm{HSD}}(\varepsilon)}{\tau_{\mathrm{HSD}}(0)} = 1 - \frac{\ln\varepsilon}{10}. \qquad (28)$$

Using numerical methods, we verified that the numerical value of the first-order expansion in terms of $\ln\varepsilon$ is indeed 0.1 with accuracy of about one part per million. We also found that the quadratic fit

$$\frac{\tau_{\mathrm{HSD}}(\varepsilon)}{\tau_{\mathrm{HSD}}(1)} = 1 - 0.1\ln(\varepsilon) + 0.0473\ln^2(\varepsilon), \qquad (29)$$

provided a good description of the exact collision times. This analytical relationship is depicted in fig. 3 as a solid line together with the circles that correspond to the results of numerical integration of eq. (11). The analytical relationship for collision time increase for the LSD model (eq. (25)) is depicted in fig. 3 by the dashed line for comparison. Note that for both the LSD and HSD models, the change in the collision time due to dissipation is relatively small as the dissipative force decreases both the rate of deformation and its depth. Hence the collision time is only weakly affected.

The end of the collision. – Until now we followed convention in identifying the end of the collision by the moment when the particle deformation becomes zero again. However, it has been recently highlighted that this approach implies the presence of a fictitious attractive force at the rebound [18,19]. This can also be observed at the bottom part of fig. 1 where velocity at the rebound reaches an extremum and then decreases before the deformation becomes zero. As there are no attractive forces in the model, there is an argument that the particles are expected to lose contact when the velocity reaches its maximum and the total force turns to zero [18]. Identifying the end of the collision by this event would increase the actual coefficient of restitution and will make it larger than the parameter ε entering eq. (8). The collision time, on the other hand, will be reduced.

Therefore, if one chooses to restrict the total force to be repulsive, appropriate adjustments have to be made to the input value of ε as described for the LSD model in ref. [18]. Due to the simple scaling properties described in this letter, exactly the same relationship between ε and the actual coefficient of restitution (which will now be somewhat larger than ε) will be true for the HSD model.

Model choice. – Whether to terminate the collision when the deformation turns zero ($\delta = 0$) or the force turns zero ($\ddot{\delta} = 0$) depends on ones approach to modelling the collision, as, of course, is the choice of a particular model (see ref. [4] for some examples). If the priority is to replicate the pre-defined coefficient of restitution, like in a collision operator, a phenomenological model such as the LSD or HSD can be used with a simpler $\delta = 0$ condition. However, if one wants to model a particular physical phenomenon like viscoelasticity, for example, the $\ddot{\delta} = 0$ condition should be used to avoid the non-physical attraction. For a collision of viscoelastic spheres, a model very similar to the HSD (with damping force proportional to $\sqrt{\delta}$ rather than $\sqrt[4]{\delta}$) has been widely used [12,13]. This model not only predicts lower coefficient of restitution for higher impact velocities, as often seen experimentally, but also covers other than viscoelastic dissipation mechanisms [20]. The advantage of using the HSD model is that it gives well-defined means of control over the collision properties and can be readily generalized to model more complex predefined $\varepsilon(v)$ profiles.

Conclusions. – To summarise, we have deduced a straightforward procedure for mapping the Hertzian oscillator with a viscous force proportional to the 1/4 power of the deformation onto a linear spring-dashpot model for which analytical solution is known. We demonstrated that, with appropriate rescaling of the coordinate axis, both models follow the same phase trajectory. This allowed us to express the coefficient in front of the dissipative force via the experimentally observed coefficient of restitution ε (see eq. (17)). The maximum deformation, indicative of particle's Young's modulus when measured experimentally, was shown to asymptotically change as $\sqrt{\varepsilon}$ for the LSD model (see eq. (21)) and as $\varepsilon^{2/5}$ for the HSD model.

Using the scaling properties of the HSD phase trajectory we demonstrated that the collision time is proportional to $1/v_0^{1/5}$ for any value of the restitution coefficient ε. The energy dissipation produced a more significant increase of the collision time within the HSD model (proportional to $-0.1\ln\varepsilon$ for $\varepsilon \approx 1$) when compared to the LSD model (according to eq. (25) τ is constant at first-order approximation). However, for the typical experimental values of the restitution coefficient $0.5 < \varepsilon < 1.0$, the collision time increase is only within 3% for the LSD model and 10% for the HSD model.

Finally, we have commented on the use of the alternative criterion to describe the end of the collision based on preventing the total force becoming attractive as described in detail in ref. [18]. Whichever criterion is chosen, it is straightforward to match the experimentally observed restitution coefficient to that obtained from a numerical integration. Since using the LSD or HSD model is already an approximation to a granular collision, we find that using a simpler collision criterion considered in this letter is more practical especially for lightly damped collisions, for which the difference between the two criteria becomes decreasingly small.

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