2015

Paper I

A1. (a) $(x+1)(x-1)^2$; (b) $\frac{1}{4}\left(\frac{3-x}{(x-1)^2}+\frac{1}{x+1}\right)$. 2. (a) $m=e$; (b) $(1,e)$.

3. (a) circle centre $(1,0)$ radius 2. 4. (a) $-\operatorname{cosec}^2 x$. 5. (a) $\frac{x^2}{2}\sin x^2+\frac{1}{2}\cos x^2+c$; (b) $\frac{\pi}{4}-\frac{1}{2}$.

6. (a) $y=1, x=\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2},0$; (b) $-3\cos 3t\,/\sin t$. 7. $1-\frac{3x}{2}-\frac{9x^2}{8}$; $-\frac{1}{3}<x\le\frac{1}{3}$.

8. $x=\pm\frac{\pi}{2},\pm\frac{\pi}{4},\pm\frac{3\pi}{4}$. 9. (a) $-\sqrt{6}$; (b) $y=\frac{x}{\sqrt{6}}+\frac{1}{2}$. 10. (a) 500500; (b) 4094.

B11. (a)(i) -1, $-i$; (b) $\dot z=\left(\dot r+ir\dot\theta\right)e^{i\theta}, \ddot z=\left(\ddot r-r\dot\theta^2+i\left(2\dot r\dot\theta+r\ddot\theta\right)\right)e^{i\theta}$; (c)(i) circle centre $(0,0)$, radius 2; (ii) $\dot r=\ddot r=0, \dot z=2i\dot\theta e^{i\theta}, \ddot z=\left(-2\dot\theta^2+2i\ddot\theta\right)e^{i\theta}$;

(iii) radial = $\ddot r-r\dot\theta^2$, transverse = $2\dot r\dot\theta+r\ddot\theta$.

12. (a)(i) exact, $f=e^{x+y}+c$; (ii) not exact; (iii) exact, $f=x^2y^3z^4+c$;

(b) $g(2m\pi,0)=0$ minima, $g\left((2m+1)\pi,0\right)=2$ saddles.

13. (b) $u(x,t)=(t+a)^{-\frac{1}{2}}e^{-\frac{(x+b)^2}{4\lambda(t+a)}}$; (c) $(4\lambda t+1)^{-\frac{1}{2}}\left(e^{-\frac{(x+1)^2}{4\lambda t+1}}+e^{-\frac{(x-1)^2}{4\lambda t+1}}\right), t>-\frac{1}{4\lambda}$.

14. (a) $a-a\cos t, a\sin t, \frac{\sin t}{1-\cos t}$; (c) $a\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$; (d) $3\pi a^2$; (e) $8a$.

15. (b) $\cosh\sqrt{2}+\frac{(x-2)}{2\sqrt{2}}\sinh\sqrt{2}+\frac{(x-2)^2}{16}\left(\cosh\sqrt{2}-\frac{1}{\sqrt{2}}\sinh\sqrt{2}\right)$; (c)(i) $x-2x^2+\frac{17x^3}{6}$;

(ii) $1+\frac{x^2}{3}-\frac{19x^4}{120}$.

16. (a)(i) 3/7, 4/7, 3/7; (ii) 2/7; (iii) 1/2, 1/2; (iv) 6/7; (v) 12/7; (b) $\frac{N_B}{N}$.

17. (a) $\frac{b^3}{3}$; (b)(i) $\sin^{-1}\left(\frac{x}{2}\right)+c$; (ii) $\ln\sqrt{\left(\frac{x}{2}+1\right)^2+1}-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}+1\right)+c$;

(iii) $F(k)=\pi^k-k(k-1)F(k-2)$; $F(5)=\pi^5-20\pi^3+120\pi$.

18. (d) $\lambda=1, \mathbf{v}=(1\quad 1\quad 1)^{\mathrm{T}}, \lambda=e^{\frac{2i\pi}{3}}, \mathbf{v}=\left(1\quad e^{\frac{4i\pi}{3}}\quad e^{\frac{2i\pi}{3}}\right)^{\mathrm{T}}, \lambda=e^{\frac{4i\pi}{3}}, \mathbf{v}=\left(1\quad e^{\frac{2i\pi}{3}}\quad e^{\frac{4i\pi}{3}}\right)^{\mathrm{T}}$;

(e) $\lambda=1, \mathbf{v}=(1\quad 1\quad 1)^{\mathrm{T}}, \lambda=e^{\frac{2i\pi}{3}}, \mathbf{v}=\left(1\quad e^{\frac{2i\pi}{3}}\quad e^{\frac{4i\pi}{3}}\right)^{\mathrm{T}}, \lambda=e^{\frac{4i\pi}{3}}, \mathbf{v}=\left(1\quad e^{\frac{4i\pi}{3}}\quad e^{\frac{2i\pi}{3}}\right)^{\mathrm{T}}$;

(f) $\frac{1}{2}(\mathbf{A}+\mathbf{A}^{\mathrm{T}})$; (g) $1,-\frac{1}{2},-\frac{1}{2}$.

19. (a) $\frac{3}{16}\left[4N\left(-\frac{1}{3}\right)^{N+1}-\left(-\frac{1}{3}\right)^{N}+1\right], \frac{3}{16}$.

20. (c) $-\alpha\rho\,\hat{\boldsymbol{\rho}}+2\alpha z\,\boldsymbol{k}$; (e) $\pm\alpha RA$ (depending on which way round the edge one goes).

Paper II

A1. (a) $\hat{\boldsymbol{n}}=\frac{1}{\sqrt{30}}(-5\quad 1\quad 2)^{\mathrm{T}}, \frac{2}{\sqrt{30}}$. 2. (a) $-2, 1\pm\sqrt{3}$. 3. x^2, minimum.

4. $\lambda=-1, \mathbf{v}=\frac{1}{\sqrt{2}}(1\quad -1)^{\mathrm{T}}, \lambda=3, \mathbf{v}=\frac{1}{\sqrt{2}}(1\quad 1)^{\mathrm{T}}$.

5. $\boldsymbol{F}=(\cos y^5\sinh z\quad -5y^4x\sin y^5\sinh z\quad x\cos y^5\cosh z)^{\mathrm{T}}$, curl $\boldsymbol{F}=\mathbf{0}$. 6. (a) $2\boldsymbol{r}$; (b) $3a$.

7. $y=x^2$. 8. $\frac{4}{5}\pi R^5$. 9. $\sin x+2\sin 2x$. 10. (a) $\frac{1}{2^{10}}$; (b) $1-\frac{11}{2^{10}}$.

B11. (a) $\lambda=1, \mu=-3, \mathbf{p}=(3\quad 0\quad 1)^{\mathrm{T}}$; (b)(i) the point $\boldsymbol{i}$; (ii) the line $(t,-1-t,-1-t)$;

(iii) no points.

12. (b) $a^2\pi\sin\frac{\beta}{2}$ from the centre along the line through the centre of mass.

13. (a) $\boldsymbol{F}$ is not; $\boldsymbol{G}$ is, $\phi=-x^2y\cosh z+c$; (b) both $\cosh 1$; (c) $3\cosh 1-3\sinh 1$, $\cosh 1$.

14. (b) $\frac{2}{T_2^2}$; (c) $\frac{t}{T_1}$ for $0 < t < T_1$; (d) $\frac{t^2}{T_2^2}$ for $0 < t < T_2$; (e) $\frac{t^3}{T^3}$ for $0 < t < T$;

(f) $R(t) = \frac{t}{T} + \frac{t^2}{T^2} - \frac{t^3}{T^3}$ for $0 < t < T$; (f) $\left(\frac{T}{3}, \frac{11}{27}\right)$.

15. (a)(i) no solutions; (ii) $f = e^{-2t} + 2e^{-6t} - 3e^{-4t}$; (b) $y = (1 - 2x + 2x^2)e^{-x}$, $y = 5e^{-2}$.

16. (a) $\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ fn of y only; (i) $y = kx^{-1/2} - x$; (ii) $y + y^2 \tan x = k$;

(b) $\ln\sqrt{y^2 + 2xy + 2x^2} - 2\tan^{-1}\left(\frac{y}{x} + 1\right) = k$.

17. You might get different versions depending on how you form your "eigenvector equation":

(c) $\begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \omega^2 m \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$; (d) $\lambda = 2k$, $\mathbf{v} = \frac{1}{\sqrt{2}}(1 \quad 0 \quad -1)^{\mathrm{T}}$, $\lambda = k(2 + \sqrt{2})$, $\mathbf{v} = \frac{1}{2}(1 \quad -\sqrt{2} \quad 1)^{\mathrm{T}}$, $\lambda = k(2 - \sqrt{2})$, $\mathbf{v} = \frac{1}{2}(1 \quad \sqrt{2} \quad 1)^{\mathrm{T}}$; (f) x_1 and x_3 are in phase, x_2 is opposite in phase and $\sqrt{2}$ bigger in magnitude.

18. (a) $\omega = \frac{2\pi}{T}$; (c) $a_n = 0$ if n is odd, $a_n = \frac{2V_0}{\pi(1-n^2)}$ if n is even, $b_1 = V_0/2$, $b_n = 0$ otherwise.

19. (b) $f = \ln n$; (c) $\sum_{i=1}^{n} ae^{-by_i} = 1$, $\sum_{i=1}^{n} y_i ae^{-by_i} = Y$.

20. (a) $u_s = \frac{G}{n^2\pi^2\nu}\sin n\pi x$; (b) $\frac{\partial\tilde{u}}{\partial t} = \nu\frac{\partial^2\tilde{u}}{\partial x^2}$, $\tilde{u}(0,t) = 0$, $\tilde{u}(1,t) = 0$, $\tilde{u}(x,0) = \frac{-G}{n^2\pi^2\nu}\sin n\pi x$;

(c) $\tilde{u} = \frac{-G}{n^2\pi^2\nu}e^{-n^2\pi^2\nu t}\sin n\pi x$;

(d) $u(x,t) = \frac{G}{n^2\pi^2\nu}\sin n\pi x\left(1 - e^{-n^2\pi^2\nu t}\right)$, $Q(t) = \frac{G\left(1-e^{-n^2\pi^2\nu t}\right)(1-\cos n\pi)}{n^3\pi^3\nu}$.

2016

Paper I

A1. (a) $(a+b)(a^2-ab+b^2)$; (b) $x=2$. 2. (a) $-\frac{1}{2}(x-1)^2+\frac{1}{2}$; (b) $\frac{\frac{3}{2}}{x-1}-\frac{\frac{1}{2}}{x+1}$.
3. (a) 10^8-4; (b) 0. 4. $\left[\frac{4\pi}{3},\frac{5\pi}{3}\right]\pm 2n\pi$. 5. (a) $x=\pm\frac{1}{\sqrt{2}}$. 6. (a) $y=(2-x)^{-1}$.
7. (a) $-\frac{1}{3}\cot(3x+1)+c$; (b) $\frac{(n-2)^{n+1}}{n+1}-\frac{(-2)^{n+1}}{n+1}$. 8. $-\sin x\,(\ln 2+1)(2e)^{\cos x}$. 9. -. 10. $0\le k\le\frac{4}{3}$.
B11. (a) $y=0$; (b) $z=i$; (c) $b=\frac{(\lambda^2+1)i}{(\lambda^2-1)}, c=1$, centre is $\left(0,-\frac{(\lambda^2+1)}{(\lambda^2-1)}\right)$, radius is $\frac{2\lambda}{|\lambda^2-1|}$;
(d)(i) $|b|^2=c$; (ii) $|b|^2>c$; (iii) impossible.
12. (b)(i) Min at (0,0), saddles at $(\pm1,\pm1)$.
13. (a) $\nabla.\mathbf{v_1}=0, \nabla.\mathbf{v_2}=10y$; (b) $I_1=\frac{5\pi}{2}bc(a_zb-a_x)+2bc(a_x-a_y), I_2=b^3+25\pi^2c^2b/4$;
(c) $\mathbf{v_1}$ is not, $\mathbf{v_2}$ is conservative; (d) $\phi=\mathrm{x}^2y+y^3+z^2y+c$.
14. (b)(i) $\lambda=1,(0\quad 0\quad 1)^T,\lambda=2,\frac{1}{\sqrt{2}}(1\quad 1\quad 0)^T,\lambda=4,\frac{1}{\sqrt{2}}(1\quad -1\quad 0)^T$; (ii) $\frac{1}{8}\begin{pmatrix}3&1&0\\1&3&0\\0&0&8\end{pmatrix}$.
15. (a)(i) $\frac{1}{2}\ln 2+\frac{x}{4}(\ln 2+1)+\frac{x^2}{8}\left(\ln 2+\frac{1}{2}\right)$; (ii) $x-\frac{x^3}{3}+\frac{x^5}{5}$; (iii) $\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{45}$;
(b)(i) $\frac{d^2g}{df^2}=-\frac{d^2f}{dg^2}/\left(\frac{df}{dg}\right)^3, \frac{d^3g}{df^3}=3\left(\frac{d^2f}{dg^2}\right)^2/\left(\frac{df}{dg}\right)^5-\frac{d^3f}{dg^3}/\left(\frac{df}{dg}\right)^4$;
(ii) $b_1=1/a_1, b_2=-a_2/a_1^3, b_3=2a_2^2/a_1^5-\ a_3/a_1^4$.
16. (a)(i) $(R-r)/R$; (v) $r^2/2R$; (vi) $r^3/3R\ -r^4/4R^2$; (b)(i) $(r-t)/R$; (ii) $(R-r)/r$; (iii) t/R.
17. (a)(i) $y=a_1x+a_0$; (ii) $\sum_{i=0}^{n-1}a_ix^i$; (b)(i) $A_0+A_1e^x+A_2e^{-x}+A_3\cos x+A_4\sin x$; (ii) $-\frac{1}{2}x^2$;
(iv) $y=1+\frac{\cosh x}{2}-\frac{1}{2}\cos x-\frac{x^2}{2}$.
18. (a)(iii) $I=\frac{1}{\alpha}$; (b)(ii) $\frac{(m-2)!(n-2)!}{(m+n-3)!}$.
19. (a) & (b) $\begin{pmatrix}\tan\theta\cos\phi & \tan\theta\sin\phi & 1\\ W\sec^2\theta\, cos\phi & W\sec^2\theta\sin\phi & 0\\ -\mathrm{W}\tan\theta\sin\phi & \mathrm{W}\tan\theta\cos\phi & 0\end{pmatrix}$; (c) $\frac{1}{W}\begin{pmatrix}0 & \cos^2\theta\cos\phi & -\cot\theta\sin\phi\\ 0 & \cos^2\theta\sin\phi & -\cot\theta\cos\phi\\ W & \cos^2\theta\tan\theta & 0\end{pmatrix}$.
20. (a)(ii) Yes; (iii) $(y^2\quad z^2\quad x^2)^T$; (b)(ii) $\pi a^3b/4$; (c) $\pi a^3b/4$.

Paper II

A1. $x\ge1, y=0$. 2. $\boldsymbol{r_0}=(1\quad 2\quad -1)^T, t=(4\quad 3\quad 2)^T$. 3. $\frac{\partial f}{\partial r}=\cos 2\theta\frac{\partial f}{\partial x}+\sin 2\theta\frac{\partial f}{\partial y}$,
$\frac{\partial f}{\partial\theta}=-2\mathrm{r}\sin 2\theta\frac{\partial f}{\partial x}+2\mathrm{r}\cos 2\theta\frac{\partial f}{\partial y}$. 4(i) $\left(\frac{35}{36}\right)^2\frac{1}{36}$; (ii) $\frac{36}{71}$. 5. (a) Yes; (b) No. 6. (a) Zero; (b) $\sin\frac{\pi}{8}\cos\frac{\pi}{4}$.
7. (a) $\sin x+\sin 2x+\sin 3x$; (b) $\cos x+2\cos 2x+3\cos 3x$. 8. $y=xe^x$.
9. $\frac{4}{3}\pi(b^3-a^3)+4\pi\left(\frac{1}{c}-\frac{1}{d}\right)$. 10. (a) xy; (b) saddle.
B11. (a)(i) $\mathbf{r}.(\mathbf{a}\wedge\mathbf{b})=0$; (ii) $\mathbf{r}.(\mathbf{a}\wedge\mathbf{b})>0$; (c) $\mathbf{r}.(\mathbf{a}\wedge\mathbf{b})>0, \mathbf{r}.(\mathbf{b}\wedge\mathbf{c})>0, \mathbf{r}.(\mathbf{c}\wedge\mathbf{a})>0$,
$\mathbf{r}.(\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a})<\mathbf{a}.(\mathbf{b}\wedge\mathbf{c})$; (d) $\sqrt{6}$.
12. (a) $\left(r,\theta,a(1-r^2/b^2)\right)$; (b) $\alpha ab^2/4$; (d) $\alpha b^2/2$;
(e) $\mathbf{S}_{ABCA}=\left(\left(\frac{2ab}{3}\right)\sin\alpha\quad\left(\frac{2ab}{3}\right)(1-\cos\alpha)\quad\alpha b^2/2\right)^T$.
13. (b)(i) $t=0$ to infinity; (ii) $3a/2$; (c) $y^3+x^3-3ayx=0, x=y$.
14. (a) $y=\frac{1}{3}(a^2+x^2)+c(a^2+x^2)^{-\frac{1}{2}}$; (b)(i) $|x+y-2|=Ae^{-\frac{1}{2}(x+3y)}$; (ii) Yes, $y=2-x$;
(c)(ii) $x=\frac{B|1-p^2|}{p^2}$; (iii) $y^2=2Bx+B^2$.

15. (b) $x = \frac{t-s}{t-1}, y = \frac{s+t-2}{t-1}, z = -1$

$t \neq 1$ one solution, $t = 1, s = 1$ infinitely many solutions, $t = 1, s = 6$ no solutions.

16. (a)(i) $0{:}\frac{1}{10}, 1{:}\frac{3}{5}, 2{:}\frac{3}{10}$; (ii) $\mu = \frac{6}{5}, \sigma = \frac{3}{5}$; (iii) $\frac{3}{5}$; (iv) $\frac{7}{10}$; (b)(i) $\frac{13}{30}$; (ii) $\frac{12}{13}$.

17. (b)(i) $\mathbf{v} = (2y, -x)$; (v) $n = m = 2, A = \frac{1}{2}, f(x^2 + 2y^2) + \frac{1}{2}x^2y^2$;

(vi) $\frac{1}{2}(x^2 + 2y^2 - 2) + \frac{1}{2}x^2y^2$.

18. (b)(i) Fourier coeffs are sin: $b_n \cos\frac{2\pi nl}{L}$ and cos: $-b_n \sin\frac{2\pi nl}{L}$; (c)(i) $-\frac{K}{n\pi}(-1)^n$;

(ii) $\sum_{n=1}^{\infty} \frac{K(-1)^n}{n\pi} \sin\frac{2\pi nvt}{L} \cos\frac{2\pi nx}{L} - \frac{K(-1)^n}{n\pi} \cos\frac{2\pi nvt}{L} \sin\frac{2\pi nx}{L}$.

19. (a) a cube; (c)(i) $\frac{1}{2}$; (ii) $\cos a$; (iii) e^{2a}.

20. (b) $f = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct)$, $f = (Ax + B)(Ct + D)$,

$f = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda ct} + De^{-\lambda ct}\right)$; (c) $f = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)$;

(d) $f = \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}$.

<u>2017</u>

Paper I

A1. (a) $\ln 2$; (b) $y=2e^{-2}$. 2. (a) $-\frac{2}{3}\cos^3 x+c$; (b) $\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$.
3. (a) $2^x \ln 2$; (b) $-\tan^2 t$. 4. (a) $-\frac{2}{3}$; (b) $y=-\frac{2}{3}x+\frac{5}{3}$.
5. (a) $x=2$; (b) $(x-2)(2x+1)(x+3)$. 6. (a) $(1+2x)^4$.
7. (a) $\sin^2\frac{\pi}{12}=\frac{1}{2}\left(1-\sqrt{1-\sin^2\frac{\pi}{6}}\right)$; (b) 0.067 . 8. (a) $x=\frac{\pi}{2}$ or $x=\frac{\pi}{6}$; (b) $x=\frac{3\pi}{2}$.
9. (a) $y=\frac{1}{c-x}$; (b) $y=\frac{1}{2-x}$. 10. (a) $OA=5$, $OB=13$; (b) $63/65$.
B11. (a) $z=re^{i\theta}$; (b) (i) $\Re=\frac{1}{2}\ln\left(x^2+y^2\right)$, $\Im=\tan^{-1}\frac{y}{x}+2n\pi$; (ii) $\Re=\frac{x}{x^2+y^2}$, $\Im=\frac{-y}{x^2+y^2}$;
(iii) $\Re=0$, $\Im=2y\left(x^2+y^2\right)$; (iv) $\Re=\cosh x\cos y$, $\Im=\sinh x\sin y$;
(c) (i) circles centred on origin; (ii) circles just touching the origin, centres on x-axis;
(d) $x=\cosh^{-1}2$, $y=(2n+1)\pi$, n integer.
12. (a)(ii) $\left(\frac{\partial x}{\partial y}\right)_z=\frac{xz}{\cosh(x+z)-yz}$, $\left(\frac{\partial x}{\partial z}\right)_y=\frac{xy-\cosh(x+z)}{\cosh(x+z)-yz}$, $\left(\frac{\partial y}{\partial z}\right)_x=\frac{xz}{\cosh(x+z)-xy}$;
(b) All a , b not both zero; (ii) $f=\tan^{-1}\left(\frac{by}{ax}\right)+c=\tan^{-1}\left(\frac{ax}{by}\right)+k$; (iii) $y=Kx$.
13. (a) $y=ke^{-2x}+x^2-x+c$; (b) $y=e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$; $y=e^{-3x^2/2}\left(3\left(x^2+1\right)^{3/2}-2\right)$.
14. (b)(i) Minimum at $(0,0)$, saddles at $\left(\frac{\pm 1}{\sqrt{(-a)}},0\right)$ if $a<0$; (iii) $y=\frac{\pm 1}{\sqrt{2}}\left(x^2-1\right)$.
15. (b) $9^{1/3}\approx 2+\frac{1}{12}-\frac{1}{(18)(16)}$, $\max R_n=\frac{5}{81(256)}$; (c)(i) $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4$;
(ii) $1-\frac{x^2}{2}-x^3+\frac{5x^4}{24}$
16. (a) $N=(13)(48)$; (b) Binomial $\binom{100}{n}p^n(1-p)^{100-n}$, $\sum_{n=51}^{100}\binom{100}{n}p^n(1-p)^{100-n}$;
(c) (ii) $\frac{1}{2}\tanh\left(\frac{\mu}{2s}\right)$.
17. (a) (i) $\ln(\ln x)+c$; (ii) $\cosh x+\cosh^{-1}x+c$; (b) $I_2=\frac{3\sqrt{\pi}}{4}$, $I_3=\frac{15\sqrt{\pi}}{8}$.
18. (a) $\boldsymbol{A}\boldsymbol{A}^T=\frac{1}{2}\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}$, $\boldsymbol{A}^T\boldsymbol{A}=\frac{1}{3}\begin{pmatrix}4 & 1 & -2\\ 1 & 1 & 1\\ -2 & 1 & 4\end{pmatrix}$; (b) $\lambda=1$ $\boldsymbol{u}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$,

$\lambda=-1$ $\boldsymbol{u}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$; (c) $\sigma=0$ $\boldsymbol{v}=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $\sigma=1$ $\boldsymbol{v}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$,

$\sigma=2$ $\boldsymbol{v}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$; (d) $\boldsymbol{0}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}1\\-1\end{pmatrix}$.
19. (a) ; (d) (i) diverges (ii) converges (using appropriate integral approximations is the least horrid method).
20. (a) $y^3=x^3+cx$; (b) $x^2+2y^2=k$.

Paper II

A1. (a) $90°$; (b) $\begin{pmatrix}1 & 0 & 2\end{pmatrix}^T$. 2. (a) $\Re = e^{-y}(x\cos x - y\sin x)$, $\Im = e^{-y}(x\sin x + y\cos x)$.
3. (a) $\lambda = \pm a$; (b) $\lambda = a$: $\boldsymbol{u} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $\lambda = -a$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$. 4. $x^3 - x^5$. 5. $\frac{3}{8} + \frac{1}{2}\cos 2x + \ldots$
6. $\begin{pmatrix}\sin(y-z) & \sin(z-x) & \sin(x-y)\end{pmatrix}^T$. 7. (a) $y = A\cos 3x + B\sin 3x$; (b) $y = \cos 4x$.
8. (a) $\mathbf{0}$; (b) $\frac{1}{12}$. 9. (a) $u' y^{-\frac{1}{2}}$; (b) $u'' y^{-1} - \frac{1}{2} u' y^{-\frac{3}{2}}$. 10. (a) $\frac{(p+q)}{2}$; (b) $\frac{P}{(p+q)}$.
B11. (b) $|\boldsymbol{a} - \boldsymbol{b}| < p + q$ and $|\boldsymbol{a} - \boldsymbol{b}| > |q - p|$.
12. (a)(ii) $4\pi\rho_0 h_0 (R^2 + 2Rh_0 + 2h_0^2)$; (b)(ii) $A = 1, B = \frac{1}{3}$.
13. (d) $\phi = x^2 y - y^2 - xz^3 - z + c$; (e) -126 .
14. (a) $\mu = \sum_{x=0}^{\infty} xP(x)$, $\sigma^2 = \sum_{x=0}^{\infty} x^2 P(x) - \mu^2$; (b)(ii) $\sum_{x=0}^{K} \frac{e^{-\lambda}\lambda^x}{x!}$; (c) (i) $(1 - e^{\lambda})^{10}$; (iii) $50\lambda^2 e^{-10\lambda}$.
15. (a)(i) $\mu = y^{-3}$, $x^2 - xy + ky^2 = 0$; (ii) $\mu = \cos x$, $y\cos^2 x + y^4 \cos^3 x = c$;
(b) $y = 2x^{\frac{1}{2}} + x + c$ if $x > 0$, or $y = Ax + B$.
16. (b)(i) $-\frac{46}{3}$; (iii) $-\frac{46}{3}$.
17. (a) $\frac{1}{2}\begin{pmatrix}3 & -1 & 0\\-1 & 3 & 0\\0 & 0 & 8\end{pmatrix}$ $\lambda = 1$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\lambda = 2$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\lambda = 4$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$; (b) $\boldsymbol{y} = K\begin{pmatrix}1 & 1 & 0\end{pmatrix}^T e^{-t}$,
$\boldsymbol{y} = K\begin{pmatrix}1 & -1 & 0\end{pmatrix}^T e^{-2t}$, $\boldsymbol{y} = K\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T e^{-4t}$; (c) $\boldsymbol{y}_0 = K^{-1}\boldsymbol{f}_0$, $\boldsymbol{y}_0 = \begin{pmatrix}1 & 1 & 1/4\end{pmatrix}^T$;
(d) $\boldsymbol{y} = \begin{pmatrix}1 & 1 & 0\end{pmatrix}^T e^{-t} + \frac{1}{4}\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T e^{-4t}$.
18. (a) $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$; (b) $p = 3$.
19. (b) $V_p = \frac{R^2 H}{2}$; (c) the cylinder.
20. (a)(i) $y = \left(1 - \frac{1}{2}\ln(1+x)\right)^2$; (ii) $y = (1 - 3x)^{-\frac{1}{3}} e^{-\frac{x}{3}}$; (b) $T = Ce^{-\left(1 + \frac{n^2\pi^2}{L^2}\right)t}$; $X = B\sin\left(\frac{n\pi x}{L}\right)$ so general solution is $u(x, t) = \sum_{n=1}^{\infty} B_n e^{-\left(1 + \frac{n^2\pi^2}{L^2}\right)t} \sin\left(\frac{n\pi x}{L}\right)$.