

2015

Paper I

- A1. (a) $(x+1)(x-1)^2$; (b) $\frac{1}{4}\left(\frac{3-x}{(x-1)^2} + \frac{1}{x+1}\right)$. 2. (a) $m = e$; (b) $(1, e)$.
3. (a) circle centre $(1,0)$ radius 2. 4. (a) $-\operatorname{cosec}^2 x$. 5. (a) $\frac{x^2}{2} \sin x^2 + \frac{1}{2} \cos x^2 + c$; (b) $\frac{\pi}{4} - \frac{1}{2}$.
6. (a) $y = 1, x = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$; (b) $-3 \cos 3t / \sin t$. 7. $1 - \frac{3x}{2} - \frac{9x^2}{8}$; $-\frac{1}{3} < x \leq \frac{1}{3}$.
8. $x = \pm \frac{\pi}{2}, \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$. 9. (a) $-\sqrt{6}$; (b) $y = \frac{x}{\sqrt{6}} + \frac{1}{2}$. 10. (a) 500500; (b) 4094.
- B11. (a)(i) $-1, -i$; (b) $\dot{z} = (\dot{r} + ir\dot{\theta})e^{i\theta}, \ddot{z} = (\ddot{r} - r\dot{\theta}^2 + i(2\dot{r}\dot{\theta} + r\ddot{\theta}))e^{i\theta}$; (c)(i) circle centre $(0,0)$, radius 2; (ii) $\dot{r} = \dot{r} = 0, \dot{z} = 2i\dot{\theta}e^{i\theta}, \ddot{z} = (-2\dot{\theta}^2 + 2i\ddot{\theta})e^{i\theta}$;
(iii) radial = $\ddot{r} - r\dot{\theta}^2$, transverse = $2\dot{r}\dot{\theta} + r\ddot{\theta}$.
12. (a)(i) exact, $f = e^{x+y} + c$; (ii) not exact; (iii) exact, $f = x^2y^3z^4 + c$;
(b) $g(2m\pi, 0) = 0$ minima, $g((2m+1)\pi, 0) = 2$ saddles.
13. (b) $u(x, t) = (t+a)^{-\frac{1}{2}}e^{-\frac{(x+b)^2}{4\lambda(t+a)}}$; (c) $(4\lambda t + 1)^{-\frac{1}{2}}\left(e^{-\frac{(x+1)^2}{4\lambda t + 1}} + e^{-\frac{(x-1)^2}{4\lambda t + 1}}\right), t > -\frac{1}{4\lambda}$.
14. (a) $a - a \cos t, a \sin t, \frac{\sin t}{1 - \cos t}$; (c) $a\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$; (d) $3\pi a^2$; (e) $8a$.
15. (b) $\cosh \sqrt{2} + \frac{(x-2)}{2\sqrt{2}} \sinh \sqrt{2} + \frac{(x-2)^2}{16}\left(\cosh \sqrt{2} - \frac{1}{\sqrt{2}} \sinh \sqrt{2}\right)$; (c)(i) $x - 2x^2 + \frac{17x^3}{6}$;
(ii) $1 + \frac{x^2}{3} - \frac{19x^4}{120}$.
16. (a)(i) $3/7, 4/7, 3/7$; (ii) $2/7$; (iii) $1/2, 1/2$; (iv) $6/7$; (v) $12/7$; (b) $\frac{N_B}{N}$.
17. (a) $\frac{b^3}{3}$; (b)(i) $\sin^{-1}\left(\frac{x}{2}\right) + c$; (ii) $\ln \sqrt{\left(\frac{x}{2} + 1\right)^2 + 1} - \frac{1}{2} \tan^{-1}\left(\frac{x}{2} + 1\right) + c$;
(iii) $F(k) = \pi^k - k(k-1)F(k-2)$; $F(5) = \pi^5 - 20\pi^3 + 120\pi$.
18. (d) $\lambda = 1, \mathbf{v} = (1 \ 1 \ 1)^T, \lambda = e^{\frac{2i\pi}{3}}, \mathbf{v} = \left(1 \ e^{\frac{4i\pi}{3}} \ e^{\frac{2i\pi}{3}}\right)^T, \lambda = e^{\frac{4i\pi}{3}}, \mathbf{v} = \left(1 \ e^{\frac{2i\pi}{3}} \ e^{\frac{4i\pi}{3}}\right)^T$;
(e) $\lambda = 1, \mathbf{v} = (1 \ 1 \ 1)^T, \lambda = e^{\frac{2i\pi}{3}}, \mathbf{v} = \left(1 \ e^{\frac{2i\pi}{3}} \ e^{\frac{4i\pi}{3}}\right)^T, \lambda = e^{\frac{4i\pi}{3}}, \mathbf{v} = \left(1 \ e^{\frac{4i\pi}{3}} \ e^{\frac{2i\pi}{3}}\right)^T$;
(f) $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$; (g) $1, -\frac{1}{2}, -\frac{1}{2}$.
19. (a) $\frac{3}{16}\left[4N\left(-\frac{1}{3}\right)^{N+1} - \left(-\frac{1}{3}\right)^N + 1\right], \frac{3}{16}$.
20. (c) $-\alpha\rho \hat{\mathbf{p}} + 2\alpha z \mathbf{k}$; (e) $\pm \alpha R A$ (depending on which way round the edge one goes).

Paper II

- A1. (a) $\hat{\mathbf{n}} = \frac{1}{\sqrt{30}}(-5 \ 1 \ 2)^T, \frac{2}{\sqrt{30}}$. 2. (a) $-2, 1 \pm \sqrt{3}$. 3. x^2 , minimum.
4. $\lambda = -1, \mathbf{v} = \frac{1}{\sqrt{2}}(1 \ -1)^T, \lambda = 3, \mathbf{v} = \frac{1}{\sqrt{2}}(1 \ 1)^T$.
5. $\mathbf{F} = (\cos y^5 \sinh z \ -5y^4 x \sin y^5 \sinh z \ x \cos y^5 \cosh z)^T, \operatorname{curl} \mathbf{F} = \mathbf{0}$. 6. (a) $2\mathbf{r}$; (b) $3a$.
7. $y = x^2$. 8. $\frac{4}{5}\pi R^5$. 9. $\sin x + 2 \sin 2x$. 10. (a) $\frac{1}{2^{10}}$; (b) $1 - \frac{11}{2^{10}}$.
- B11. (a) $\lambda = 1, \mu = -3, \mathbf{p} = (3 \ 0 \ 1)^T$; (b)(i) the point \mathbf{i} ; (ii) the line $(t, -1-t, -1-t)$;
(iii) no points.
12. (b) $a^2\pi \sin \frac{\beta}{2}$ from the centre along the line through the centre of mass.
13. (a) \mathbf{F} is not; \mathbf{G} is, $\phi = -x^2y \cosh z + c$; (b) both $\cosh 1$; (c) $3\cosh 1 - 3 \sinh 1, \cosh 1$.

14. (b) $\frac{2}{T_2}$; (c) $\frac{t}{T_1}$ for $0 < t < T_1$; (d) $\frac{t^2}{T_2^2}$ for $0 < t < T_2$; (e) $\frac{t^3}{T^3}$ for $0 < t < T$;

(f) $R(t) = \frac{t}{T} + \frac{t^2}{T^2} - \frac{t^3}{T^3}$ for $0 < t < T$; (f) $\left(\frac{T}{3}, \frac{11}{27}\right)$.

15. (a)(i) no solutions; (ii) $f = e^{-2t} + 2e^{-6t} - 3e^{-4t}$; (b) $y = (1 - 2x + 2x^2)e^{-x}$, $y = 5e^{-2}$.

16. (a) $\frac{1}{P} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$ fn of y only; (i) $y = kx^{-1/2} - x$; (ii) $y + y^2 \tan x = k$;

(b) $\ln \sqrt{y^2 + 2xy + 2x^2} - 2 \tan^{-1} \left(\frac{y}{x} + 1 \right) = k$.

17. You might get different versions depending on how you form your "eigenvector equation":

(c) $\begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \omega^2 m \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$; (d) $\lambda = 2k$, $\mathbf{v} = \frac{1}{\sqrt{2}}(1 \ 0 \ -1)^T$, $\lambda = k(2 + \sqrt{2})$, $\mathbf{v} =$

$\frac{1}{2}(1 \ -\sqrt{2} \ 1)^T$, $\lambda = k(2 - \sqrt{2})$, $\mathbf{v} = \frac{1}{2}(1 \ \sqrt{2} \ 1)^T$; (f) x_1 and x_3 are in phase, x_2 is opposite in phase and $\sqrt{2}$ bigger in magnitude.

18. (a) $\omega = \frac{2\pi}{T}$; (c) $a_n = 0$ if n is odd, $a_n = \frac{2V_0}{\pi(1-n^2)}$ if n is even, $b_1 = V_0/2$, $b_n = 0$ otherwise.

19. (b) $f = \ln n$; (c) $\sum_{i=1}^n a e^{-by_i} = 1$, $\sum_{i=1}^n y_i a e^{-by_i} = Y$.

20. (a) $u_s = \frac{G}{n^2 \pi^2 \nu} \sin n\pi x$; (b) $\frac{\partial \tilde{u}}{\partial t} = \nu \frac{\partial^2 \tilde{u}}{\partial x^2}$, $\tilde{u}(0, t) = 0$, $\tilde{u}(1, t) = 0$, $\tilde{u}(x, 0) = \frac{-G}{n^2 \pi^2 \nu} \sin n\pi x$;

(c) $\tilde{u} = \frac{-G}{n^2 \pi^2 \nu} e^{-n^2 \pi^2 \nu t} \sin n\pi x$;

(d) $u(x, t) = \frac{G}{n^2 \pi^2 \nu} \sin n\pi x (1 - e^{-n^2 \pi^2 \nu t})$, $Q(t) = \frac{G(1 - e^{-n^2 \pi^2 \nu t})(1 - \cos n\pi)}{n^3 \pi^3 \nu}$.

- A1. (a) $(a + b)(a^2 - ab + b^2)$; (b) $x = 2$. 2. (a) $-\frac{1}{2}(x - 1)^2 + \frac{1}{2}$; (b) $\frac{\frac{3}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$.
3. (a) $10^8 - 4$; (b) 0. 4. $\left[\frac{4\pi}{3}, \frac{5\pi}{3}\right] \pm 2n\pi$. 5. (a) $x = \pm \frac{1}{\sqrt{2}}$. 6. (a) $y = (2 - x)^{-1}$.
7. (a) $-\frac{1}{3} \cot(3x + 1) + c$; (b) $\frac{(n-2)^{n+1}}{n+1} - \frac{(-2)^{n+1}}{n+1}$. 8. $-\sin x (\ln 2 + 1)(2e)^{\cos x}$. 9. -. 10. $0 \leq k \leq \frac{4}{3}$.
- B11. (a) $y = 0$; (b) $z = i$; (c) $b = \frac{(\lambda^2+1)i}{(\lambda^2-1)}$, $c = 1$, centre is $\left(0, -\frac{(\lambda^2+1)}{(\lambda^2-1)}\right)$, radius is $\frac{2\lambda}{|\lambda^2-1|}$;
 (d)(i) $|b|^2 = c$; (ii) $|b|^2 > c$; (iii) impossible.
12. (b)(i) Min at (0,0), saddles at $(\pm 1, \pm 1)$.
13. (a) $\nabla \cdot \mathbf{v}_1 = 0$, $\nabla \cdot \mathbf{v}_2 = 10y$; (b) $I_1 = \frac{5\pi}{2} bc(a_z b - a_x) + 2bc(a_x - a_y)$, $I_2 = b^3 + 25\pi^2 c^2 b/4$;
 (c) \mathbf{v}_1 is not, \mathbf{v}_2 is conservative; (d) $\phi = x^2 y + y^3 + z^2 y + c$.
14. (b)(i) $\lambda = 1, (0 \ 0 \ 1)^T, \lambda = 2, \frac{1}{\sqrt{2}}(1 \ 1 \ 0)^T, \lambda = 4, \frac{1}{\sqrt{2}}(1 \ -1 \ 0)^T$; (ii) $\frac{1}{8} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.
15. (a)(i) $\frac{1}{2} \ln 2 + \frac{x}{4} (\ln 2 + 1) + \frac{x^2}{8} \left(\ln 2 + \frac{1}{2}\right)$; (ii) $x - \frac{x^3}{3} + \frac{x^5}{5}$; (iii) $\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45}$;
 (b)(i) $\frac{d^2 g}{df^2} = -\frac{d^2 f}{dg^2} / \left(\frac{df}{dg}\right)^3, \frac{d^3 g}{df^3} = 3 \left(\frac{d^2 f}{dg^2}\right)^2 / \left(\frac{df}{dg}\right)^5 - \frac{d^3 f}{dg^3} / \left(\frac{df}{dg}\right)^4$;
 (ii) $b_1 = 1/a_1, b_2 = -a_2/a_1^3, b_3 = 2a_2^2/a_1^5 - a_3/a_1^4$.
16. (a)(i) $(R - r)/R$; (v) $r^2/2R$; (vi) $r^3/3R - r^4/4R^2$; (b)(i) $(r - t)/R$; (ii) $(R - r)/r$; (iii) t/R .
17. (a)(i) $y = a_1 x + a_0$; (ii) $\sum_{i=0}^{n-1} a_i x^i$; (b)(i) $A_0 + A_1 e^x + A_2 e^{-x} + A_3 \cos x + A_4 \sin x$; (ii) $-\frac{1}{2} x^2$;
 (iv) $y = 1 + \frac{\cosh x}{2} - \frac{1}{2} \cos x - \frac{x^2}{2}$.
18. (a)(iii) $I = \frac{1}{\alpha}$; (b)(ii) $\frac{(m-2)!(n-2)!}{(m+n-3)!}$.
19. (a) & (b) $\begin{pmatrix} \tan \theta \cos \phi & \tan \theta \sin \phi & 1 \\ W \sec^2 \theta \cos \phi & W \sec^2 \theta \sin \phi & 0 \\ -W \tan \theta \sin \phi & W \tan \theta \cos \phi & 0 \end{pmatrix}$; (c) $\frac{1}{W} \begin{pmatrix} 0 & \cos^2 \theta \cos \phi & -\cot \theta \sin \phi \\ 0 & \cos^2 \theta \sin \phi & -\cot \theta \cos \phi \\ W & \cos^2 \theta \tan \theta & 0 \end{pmatrix}$.
20. (a)(ii) Yes; (iii) $(y^2 \ z^2 \ x^2)^T$; (b)(ii) $\pi a^3 b/4$; (c) $\pi a^3 b/4$.

Paper II

- A1. $x \geq 1, y = 0$. 2. $\mathbf{r}_0 = (1 \ 2 \ -1)^T, t = (4 \ 3 \ 2)^T$. 3. $\frac{\partial f}{\partial r} = \cos 2\theta \frac{\partial f}{\partial x} + \sin 2\theta \frac{\partial f}{\partial y}$,
 $\frac{\partial f}{\partial \theta} = -2r \sin 2\theta \frac{\partial f}{\partial x} + 2r \cos 2\theta \frac{\partial f}{\partial y}$. 4(i) $\left(\frac{35}{36}\right)^2 \frac{1}{36}$; (ii) $\frac{36}{71}$. 5. (a) Yes; (b) No. 6. (a) Zero; (b) $\sin \frac{\pi}{8} \cos \frac{\pi}{4}$.
7. (a) $\sin x + \sin 2x + \sin 3x$; (b) $\cos x + 2 \cos 2x + 3 \cos 3x$. 8. $y = x e^x$.
9. $\frac{4}{3} \pi (b^3 - a^3) + 4\pi \left(\frac{1}{c} - \frac{1}{a}\right)$. 10. (a) xy ; (b) saddle.
- B11. (a)(i) $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0$; (ii) $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b}) > 0$; (c) $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b}) > 0, \mathbf{r} \cdot (\mathbf{b} \wedge \mathbf{c}) > 0, \mathbf{r} \cdot (\mathbf{c} \wedge \mathbf{a}) > 0,$
 $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a}) < \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$; (d) $\sqrt{6}$.
12. (a) $(r, \theta, a(1 - r^2/b^2))$; (b) $\alpha ab^2/4$; (d) $\alpha b^2/2$;
 (e) $\mathbf{S}_{ABCA} = \left(\left(\frac{2ab}{3}\right) \sin \alpha \ \left(\frac{2ab}{3}\right) (1 - \cos \alpha) \ \alpha b^2/2\right)^T$.
13. (b)(i) $t = 0$ to infinity; (ii) $3a/2$; (c) $y^3 + x^3 - 3ayx = 0, x = y$.
14. (a) $y = \frac{1}{3}(a^2 + x^2) + c(a^2 + x^2)^{-\frac{1}{2}}$; (b)(i) $|x + y - 2| = A e^{-\frac{1}{2}(x+3y)}$; (ii) Yes, $y = 2 - x$;
 (c)(ii) $x = \frac{B|1-p^2|}{p^2}$; (iii) $y^2 = 2Bx + B^2$.

15. (b) $x = \frac{t-s}{t-1}, y = \frac{s+t-2}{t-1}, z = -1$

$t \neq 1$ one solution, $t = 1, s = 1$ infinitely many solutions, $t = 1, s = 6$ no solutions.

16. (a)(i) $0: \frac{1}{10}, 1: \frac{3}{5}, 2: \frac{3}{10}$; (ii) $\mu = \frac{6}{5}, \sigma = \frac{3}{5}$; (iii) $\frac{3}{5}$; (iv) $\frac{7}{10}$; (b)(i) $\frac{13}{30}$; (ii) $\frac{12}{13}$.

17. (b)(i) $\mathbf{v} = (2y, -x)$; (v) $n = m = 2, A = \frac{1}{2}, f(x^2 + 2y^2) + \frac{1}{2}x^2y^2$;

(vi) $\frac{1}{2}(x^2 + 2y^2 - 2) + \frac{1}{2}x^2y^2$.

18. (b)(i) Fourier coeffs are sin: $b_n \cos \frac{2\pi nl}{L}$ and cos: $-b_n \sin \frac{2\pi nl}{L}$; (c)(i) $-\frac{K}{n\pi}(-1)^n$;

(ii) $\sum_{n=1}^{\infty} \frac{K(-1)^n}{n\pi} \sin \frac{2\pi nvt}{L} \cos \frac{2\pi nx}{L} - \frac{K(-1)^n}{n\pi} \cos \frac{2\pi nvt}{L} \sin \frac{2\pi nx}{L}$.

19. (a) a cube; (c)(i) $\frac{1}{2}$; (ii) $\cos a$; (iii) e^{2a} .

20. (b) $f = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct), f = (Ax + B)(Ct + D),$

$f = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda ct} + De^{-\lambda ct})$; (c) $f = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{L} \right)$;

(d) $f = \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L}$.