2005

I/1 : (a) 1/3 (b) 1/6 (c) −1

2. (a) true (b) false (c) true (d) true (e) true but how to prove w/ suffix notation?
3. (a) $(1-p+Np)^{-1}$ (b)(i) 1/365 (ii) $N=254$ (c) 1.46
4. (a)(i) $1-e^{-\alpha t_0}$ (ii) Mean $= 1/\alpha$ Var $= 1/\alpha^2$ (iii) e^{-1} (b)(ii) $\alpha = 2C/Q$
5. (a) $\cos^{-1} 4/5$ (b) $\cos^{-1}(3/\sqrt{12})$ (c) $\sqrt{18}/2$ (d) $3/\sqrt{2}$
6. (a) – (b) and (c) $8\pi a^3$
7. (a) – (b) $a=0.4959$ $b=0.9366$ (i) 2611 (ii) 5912 (iii) 3.627×10^{47} (c) 2954
8. (a) 7/128 (b) $\sqrt{\pi}/12$ (c)(ii) $\ln \alpha$
9. (a) $y=\tan(\tan x + c)$ (b) $y=e^{-x}\left(\frac{1}{\alpha}+cx^{-\alpha}\right)$ and when $\alpha=0$ $y=e^{-x}(\ln x+c)$ (c) $y=1+(-x^2-c)^{-1}$
10. (a) $y=[(c-b)e^{ax}+(c-a)e^{bx}+(a-b)e^{cx}]/[(a-b)(c-a)(c-b)]$
    (b) $y = e^{ax}/(b-a)^2 - e^{bx}/(b-a)^2 + xe^{bx}/(b-a)$
11. Max at $x=1$; Min at $x=-1$ (a) zero (b) 1 (c) zero
12. (b) $\frac{1}{2}+\sum_{m=0}^{\infty}\frac{4}{(2m+1)^2\pi^2}\cos(2m+1)\pi x$

II/1 : (a)(i) No (ii) $\phi = xye^{z-x}+x^2$ (b) 69° in both cases

2. $4\pi(R^{2-r}-\varepsilon^{2-r})$
3. (a) $\alpha=1$: $x=z/5$ $y=-7z/5$; $\alpha=-8/5$: $x=-5z$ $y=5z/2$
   (b)(i) $S = I\cos\varepsilon + \frac{A}{\varepsilon}\sin\varepsilon$ (ii) $T = I\cosh\varepsilon + \frac{B}{\varepsilon}\sinh\varepsilon$
4. (a) Max $z=1$ at $(\pm1,0)$; Min $z=-1$ at $(0,\pm1)$
   (b) $x=(x_0-mc-y_0)/(1+m^2)$; $y=(mx_0-my_0+c)/(1+m^2)$
   (c) $r=(3V/8\pi)^{1/3}$ $h=2(3V/8\pi)^{1/3}$
5. (a) – (b) $\theta = \pm\pi/6+n\pi$ $n=0,1,2\ldots$ (c) –
6. (a) – (b) $\underline{r}\cdot\frac{1}{3}(1,2,2)=20/3$ (c) 3 (d) $(2,4,5)$ radius $\sqrt{7}$
7. (a) $F=U-TS$ (b) –
8. (a) exact, $f=6x^2+5xy+3y^2/2-9x-4y+c$ (c) exact, $f=-x/(x^2+y^2)$
   (b) not exact, IF $= x^{-1/2}$
9. (a) – (b) $\lambda=-1$ $(\sin 2\theta, -\cos 2\theta-1, 0)$ ie $(\sin\theta, -\cos\theta, 0)$
   $\lambda=1$ $(\sin 2\theta, -\cos 2\theta+1, 0)$ ie $(\cos\theta, \sin\theta, 0)$
   $\lambda=2$ $(0,0,1)$
10. –
11. (a) $\frac{5}{2}\ln 2-\frac{1}{2}\ln 3$ (b) 1/2 (c) 1
12. $g'=g/t$ $g=kt$.

2006

I/1 : (a) $A=0$ (b)(i) no (ii) yes eg $B=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ (c) $(I-D)^{-1}=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$

2 : (a)(i) $\lambda = 0, \pm\sqrt{2}$ $\underline{r} = \frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{2}(1,\pm\sqrt{2},1)$ (ii) R is orthogonal

$RMR^T = \begin{pmatrix}0&0&0\\0&\sqrt{2}&0\\0&0&-\sqrt{2}\end{pmatrix}$ M is not invertable (det = product of λ's = 0)

(b) $\lambda = 5$, just one soln.

3. (a)(i) $-\frac{1}{2}+i$ (ii) i (b)(i) $\pm\sqrt{3}-i$, $2i$ (ii) $\pm(\sqrt{3}+i)$

(iii) $ie^{(\pi/2+2n\pi)}$ $n=0,1,2\ldots$

4. (b) $(-2yz, 4xz^2, 0)$ (c) No, since $\underline{\nabla}\wedge\underline{F}\neq 0$ (d) $1/15$ in both cases

5. (a) $bc\underline{i}+ac\underline{j}+ab\underline{k}$ (b) $\underline{r}.(bc\underline{i}+ac\underline{j}+ab\underline{k}) = abc$ (c) $abc/\sqrt{(b^2c^2+a^2c^2+a^2b^2)}$

(d) $\lambda+\mu+\nu=1$ (e) $\lambda = b^2c^2/k$, $\mu = a^2c^2/k$, $\nu = a^2b^2/k$ where $k = b^2c^2+a^2c^2+a^2b^2$

(f) 90°.

6. (a)(ii) $(\underline{a}.\underline{b})\underline{c} - (\underline{c}.\underline{b})\underline{a}$ (b) $(3\sqrt{3}r_0^3\cos\theta_0\sin^2\theta_0)/2$

7. (a)(i) 0.05094 (ii) 0.0194 (iii) 1.05×10^{-5} (b)(i) $1/6$ (ii) $5/12$ (iii) $5/9$

8. (a) Max at $(0,1)$ $f=2$ Min at $(0,-1)$ $f=-2$ (b) $(1,-\frac{1}{2},\frac{1}{2})$

9. (a) $f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a)$

(b)(i) $x - x^2/2 + x^3/3 - x^4/4$ (ii) $\pi/4 + x/2 - x^2/4 + x^3/12 + \ldots$ (iii) $1 - x/2 + x^2/6 + \ldots$ yuk

10. (a) $y = \frac{1}{3}(\cos x - \cos 2x)$ (b) $y = -\frac{1}{2}\cos 2x + x/8 \sin 2x + 1/2$

(c) $y = (\frac{1}{2}x^2 + \frac{1}{6}x^3)e^x$

11. (b) $\sum_{m=0}[8/\pi(4-(2m+1)^2)]\cos(2m+1)x$

12. (b) $f=0$ everywhere

II/1 : (a) $(1+x^2)^{-1}$ (b) $x^2/18$ (c) $(32/15)/3^{1/4}$

1: (c) Show that $(CBA)_{ij} = -\delta_{ij}$ ie $\underline{s}''' = -\underline{s}$

2. (b) $4(\underline{a}.\underline{b})(b_i a_j - a_i b_j)$

3. (a) $\underline{F}_1$ and $\underline{F}_3$ (b) π^4+2 (c) 2

4. (a) $\underline{F} = \text{grad} f/(\partial f/\partial z)$ (b) $((2x(1+y), x^2+2yz, y^2)/y^2)\,dx\,dy$

(c) $19\pi a^3/10$

5. (a) (b) $\pi a^3(1-\cos\theta_0)^3\sin^3\theta_0\,d\theta$

6. (a) $\frac{2}{x}\cos x^4 - \frac{1}{x}\cos x^3$ (b) bounds are $1-\ln 2$ and 1 (c) $x\cos x + xy^2 = c$

7. (a) $x^2y^2+4x-6y=c$ (b) $(y^2+x-1)e^x = c$

(b)(i) 3.5 (ii) $15\frac{1}{6}$ (iii) $2\frac{11}{12}$

8. (a)(ii) all $\frac{1}{4^4}:(1,12,54,\underline{\underline{108}},81)$

(c)(i) $\alpha = 1/4$ (ii) $2.58\dot{3}$

9. (b) $z = e^{a(x+c)} - e^{b(x+c)}$ (c) $y = \frac{1}{x+c}+a$ $z = (x+c)e^{ax+d}$ c−x for z (can also get expressions if you treat log integral differently)

(d) $z'' - (a+b)z' + abz = 0$

10. (b) $\lim_{n\to\infty} w_n = 0$

11. Yuk

12. Max is at $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$; Min at $-(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

2007

I/A 1. $x = 1, 1/2, -2$ 2. (a) $-5\sin 5x$ (b) $2xe^x + x^2e^x$ 3(a) $(3+2x)^{1/2} + c$
3(b) $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$ 4. radius 2 centre $(0,-3)$ 5(a) 10201 (b) 3/2
6. 6 7. $\pi/2$ or $\pi/4$ 8. 30° 9. 5 10. – 11. $k = 2e$ $x = 1/2$ $y = e$

B1. –
2 (a) $Re = \pm 1/\sqrt{2}$ $Im = \pm \frac{1}{\sqrt{2}}$; $Re = 1$ $Im = \pi$; $Re = \cosh 2$ $Im = \sin\ln 2$; $Re = 0$, $Im = -1$
(b) $\sin 3x = 3\sin x - 4\sin^3 x$ $x = 0, \pi, \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$
(c) $-b \pm \sqrt{b^2-1}$ • roots trace out unit circle, centred on origin
3. (a)(i) $x - \frac{1}{3}x^3$ (ii) $-x\ln a + \frac{x^2}{2}\ln^2 a$ (iii) $-x^2 - x^4/6$ (b) died of boredom
4. (a) – (b) 0.31, 0.0362.
5 (a) (i) exact: $(1+e^y)\sin x = c$ (ii) Not exact, $\mu = \frac{1}{y}$: $\frac{x^3}{3} + x\ln y = c$
(b) $y = (1+x)^5/2 + (1+x)^3$
6. $(0,0)$, $(\pi/2, \pi/2)$, (π, π), $(0, \pi)$, $(\pi, 0)$
Min Saddle Min Max Max
7. (i) $\frac{1}{4}(e^4 - 1)$ (ii) Mass $= \pi k/4$ Area $= 4\pi(1 - 1/\sqrt{2})$
8. (a)(i) False (ii) True, $a+b$ (iii) True, ab (b)(i) 0, 1 (ii) 0, 1
(c) $-1, 0, 3$; $(1\ 0\ -1)^T$.
9. (a) $\sum 1/n\ln n$ diverges ; $\sum n\ln^2 n$ converges ; $\ln^{-1}(N+1) < R_N < \ln^{-1}(N)$
(b) converges ; diverges.
10. I'd avoid ever trying...

II/A 1. $(-2, 0, 2)$ 2. (a) zero (b) $1/2$ 3. (b) $(-1, -1)$ 4. (a) $-(6x, 6y, 2z)$
4(b) -14 (c) $\underline{0}$ 5. (a) 5 (b) $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ 6. Mod $= \sqrt{8}$ Arg $= 5\pi/12$
7. $x - x^2/2$ 8. $t = 1/(kx_0)$ 9. (a) 4/7 (b) 6/7.

B1. $r = 2\cos\theta$; $r = 2/\cos\theta$
2. –
3. (a) $A = 1 - \rho$; ρ^{n+1} ; ρ^{n-m} (b) $B = \lambda$; $e^{\lambda(y-x)}$; $\lambda e^{\lambda(y-x)}$; $1/\lambda^2$
4. (a) $y = (A + Bx)e^{-2x} + \frac{1}{16}e^{2x}$ (b) $y = \frac{-5}{4}e^{-3x}\sin 4x + \sin 5x$
5. (a) $(\partial u/\partial y)_x = (\partial v/\partial x)_y$ (b) $(\partial v/\partial x)_y$; both sides equal to $-2T/3V$.
6. (a) $\underline{0}$ (b) $\phi = -(x^2y + xz + y^2 + c)$ (c) 3 (d) $4\pi b^2 a^3/15$
7. (a)(i) True (ii) False (iii) True (b) $C^{16} = C$ (c) $\mu \neq 0$ $M^{-1} = \frac{1}{2\mu}\begin{pmatrix} 1 & \mu & -1 \\ \mu & -\mu^2 & \mu \\ -1 & \mu & 1 \end{pmatrix}$
(d)(i) $x = 1$ $y = 1 - \mu$ $z = 1$ (ii) $y = 1$ $x + z = 2$
8. $$f(x) = \underbrace{\frac{1}{2} + \sum_{n=1}^{\infty}\Big(\frac{2}{n^2\pi^2}((-1)^n - 1)\cos n\pi x}_{f_e} - \underbrace{\frac{2}{n\pi}(-1)^n \sin n\pi x\Big)}_{f_o}$$
9. Stationary pt. is $(1,1)$; contour parallel to constraint path at stationary point.
10. $\frac{dT}{dt} = a\nabla T$; $u = e^{-\pi^2\nu t}\sin \pi y$; $u = e^{-\pi^2\nu t}\sin\pi y + \frac{1}{10}e^{-9\pi^2\nu t}\sin 3\pi y$

2008 (Question numbering for online version)

I/A 1. $\theta = 75^\circ$ or 345° 2. $2x\cos x^2$; $\frac{e^{2x}}{x}(2 - 1/x)$ 3. $(x^2+1)^{1/2}+c$, $(x-1)e^x+c$

4. $x = \pm 1, \pm\sqrt{2}$ 5. $\sqrt{10}$ 6. $y = 3-x$ 8. $x<1$ or $x>2$ 9. $\sin\theta + \theta$

10. (i) $(N+1)^3$ (ii) $\frac{1}{3}((N+1)^3 - (N+1))$

B11. $\underline{n} = \frac{1}{\sqrt{3}}(\underline{i}+\underline{j}+\underline{k})$; $\underline{e}_1 = \frac{1}{\sqrt{6}}(1,1,-2)$ $\underline{e}_2 = \frac{1}{\sqrt{2}}(1,-1,0)$ ratio $= \sqrt{3}$.

coords $\frac{1}{\sqrt{6}}(x+y-2z)$, $\frac{1}{\sqrt{2}}(x-y)$

12. (b)(i) $\frac{1-x^2-y^2}{(1+x)^2+y^2}$, $\frac{-2y}{(1+x)^2+y^2}$ (ii) $e^{-y}\cos x$, $e^{-y}\sin x$ (iii) Re: $x\sin x\cosh y - y\cos x\sinh y$.
Im: $y\sin x\cosh y + x\cos x\sinh y$

(c)(i) $e^{i\theta}$ where $\theta = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$ (ii) $i(\cosh 2 \pm \sinh 2)$

13. (a) $2(x + x^3/3 + x^5/5 + \dots)$ (b) $(-19/216)x^4$ (c) $a = 2/3$ $b = 1/6$

14. (a) $1/6(17^{3/2}-1)$ (b) volume is $(p-q)\pi a^3$; area is $2(p-q)\pi a^2$.

15. (b) $P(\alpha < z < \beta) = \int_\alpha^\beta \sum_{n=-\infty}^{\infty} p_n f(z-n)\,dz$; density $= \sum_{n=-\infty}^{\infty} p_n f(z-n)$

16(a)(i) $y = 1 - x^{-3}$ (ii) $y = \frac{1}{2}\ln(1+x^2)$ (iii) $y = x\sin(\ln x + c)$

(b)(i) $y = 2\sin x + 2e^{-x}\cos x$ (ii) $y = -1/x$??

17. (a)(ii) $(0, 1/4)$, $(1/4, -1/4)$, $(-1/4, 0)$ (iii) $(1/3, -1/3)$, minimum

(b) saddle at $(0,0)$; minimum at $(-2/3, 0)$

18. (a)(i) -4 (ii) $\frac{1}{4}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 0 & 2 \\ 3 & 2 & -3 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (b) – (c) $\mu = 5$.

19. (a) (i) all $n \geq 0$ (ii) $n = 1, 2, 3$ (iii) $n = 2, 3$ (b) $\max(A, B)$ (c) nothing; converges.

20. $V_x \geq V_t$.

II/A 1. $\pi/3$ 2. (i) 1 (ii) 0 3. $x = \pm y$ 4. (i) $2r$ (ii) $5r^2$ 5. $(0,1)$ 6. $\pm 3/2$

7. $e - ex^2/2!$ 8. (i) $\alpha = 1$ (ii) $u = x^2 - x + t$ 9. (i) $\pi/4$ (ii) $8r\,dr$

B10. –

11. (a)(i) $\frac{L!(N-5)!}{(L-5)!\,N!}$ (ii) $\frac{5!}{n!(5-n)!}\frac{L!(N-L)!(N-5)!}{(L-n)!(N-L-5+n)!\,N!}$ (b) $\frac{L}{N}$ (c) $\frac{K!}{k!(K-k)!}f^k(1-f)^{K-k}$

(d)(i) $1/4$ (ii) $1/100$ (iii) $0.1, 0.735$

12. $I(a,b) = \sqrt{2\pi/a}\,e^{-\sqrt{ab}}$ (checked using online definite integrator!) $J(a,b) = I(b,a)$

13. (i) $y = -16e^{3x} + 9e^{4x} + 12x + 7$ (ii) $y = 5e^{-x} - 2e^{-2x} + \sin x - 3\cos x$

(iii) $y = 4x^2e^{-x}$

14. (a) $v = 4$ $A = -2$ (b) – $F_1: -1$ $F_2: -50$ (ii) $F_1: -1$ $F_2: -323/7$

15. (a)(i) is (ii) is not (b)(i) $F_1: -1$ … $\phi = x^3y - y^2z^2 + c$

16. (a)(ii) A^2 (iii) A (iv) A^2 (b) $\mu = 4$ $\lambda = 1: \frac{1}{\sqrt{5}}(-2, 0, 1)$ $\lambda = -4: \frac{1}{\sqrt{45}}(2, -5, 4)$

$\lambda = 5$: $1/3(1, 2, 2)$ (ii) $|\mu| < 1$

17. (a) $2L$ (b) – (c) $\sum(-2(-1)^n \sin nx)/n$

18. (a) $4\pi a^2 b$ (b) $4\pi a^2 b\,k$

19. (a) $a_n = \frac{2L}{n\pi c}\int_0^L v(x)\sin\frac{n\pi x}{L}\,dx$ frequencies $n=1$ and $n=3$ ratio of amplitudes 3:1.

(b) $f(x) = A\cos kx + B\sin kx$; $V(x,y) = -\pi\sum_0 \cos x\, e^{-|y|}$

2009

Paper I

A1. $-1 \pm \sqrt{3}$. 2. $x = 0$ or 9. 3. (a) $3/x$; (b) $\cos 2x - 2x\sin 2x$.

4. (a) $\frac{1}{4}(ln|x-4| - ln|x|) + c$; (b) $\frac{1}{2}e^{x^2} + c$. 5. $x = \frac{\pi}{2}$ or $\frac{2\pi}{3}$. 6. $x = 25$. 7. (a) 15352 (b) 4/5. 8. –

9. $16/\sqrt{5}$. 10 (b) -2.

B11. (a) (i) $\frac{-11x^3-6x^2}{(3x-2)^5}$; (ii) $a/(a^2-x^2)$; (c) (i) $\pi(h-x)^2 x\tan^2\alpha$; (ii) $\frac{3}{4}\cot\alpha$.

12. (a) (i) $\binom{n}{k}p^k(1-p)^{n-k}$; (ii) $np, np(1-p)$; (b) (i) $\binom{n-1}{c-1}p^c(1-p)^{n-c}$.

13. (a) $3 \pm i, \binom{2}{-i-1}, \binom{2}{i-1}, B = \begin{pmatrix} 2 & 2 \\ -i-1 & i-1 \end{pmatrix}$; (b) $-4, 1, 6, \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$.

14. (b) $e^{i\theta}$, where $\theta = \frac{\pi}{8} + \frac{n\pi}{2}, n = 0,1,\ldots$; If you regard the square root of *i* as only having one value (rather than being +/- that value) then you will get $\theta = \frac{\pi}{8} + n\pi$.

15. –

16. (b) 24; (d) $\pi, \sqrt{\pi}$.

17. (a) (i) $\sin\left(\frac{y}{x}\right) = \frac{2}{\pi x}$; (ii) $x = \tan\frac{(x+y)}{2}$ (b) $y = \operatorname{cosec}^2 x$.

18. (a) $(\frac{\partial u}{\partial s})_t + (\frac{\partial u}{\partial t})_s(\frac{\partial t}{\partial s})_v$; (b) $\mu = Ax^{-3}, f = -x^{-2}\sin y + c$.

19. (c) $\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}(C_n\cos\frac{n\pi ct}{L} + D_n\sin\frac{n\pi ct}{L})$.

20. (a) (i) No; (ii) Yes; (iii) No; (b) (i) 3; (ii) 4; (iii) 1/90.

Paper II

A1. - 2. Both zero 3. $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ 4. $\alpha = \pm 1$ 5. Max at $x = 0$, Min at $x = 2$ 6. –

7. $y = x - 1$ 8. Ignoring their (apparently daft) suggestion and finding the Taylor Series directly gives $\ln 2 + \frac{x}{2} + \frac{x^2}{8}$. 9. 0 to $\frac{\pi}{4}, \frac{3\pi}{4}$ to $\frac{5\pi}{4}, \frac{7\pi}{4}$ to 2π.

B11. (a) $(1,2,4) + \lambda(1,2,-2)$ or $x = 1+\lambda, y = 2+2\lambda, z = 4-2\lambda$;

(b) $(1,-1,0) + z(2,3,1)$; (c) $\sqrt{\frac{35}{49}}$.

12. (ii) $y = Ae^{\frac{t\left(-\lambda+\sqrt{\lambda^2-4k}\right)}{2}} + Be^{\frac{t\left(-\lambda-\sqrt{\lambda^2-4k}\right)}{2}}$, oscilliatory if $\lambda^2 < 4k$; (ii) $\omega = \sqrt{\frac{4k-\lambda^2}{2}}, t = \frac{2}{\lambda}\ln\frac{4}{3}$; (iii) $\frac{1}{(1+\omega^2)^2}\sqrt{16 + \left(\frac{2-2\omega^2}{\omega}\right)^2}$???

13. (a) (1,0) saddle, (0,1) min, (0,-1) max, (2,1) max, (2-1) min; (b) (i) $\frac{1}{\sqrt{2}}(1,0,-1)$;

(ii) $\frac{u}{\sqrt{2}}(-1,0,1)$; (iii) (0,1).

14. (a) (i) -4; (ii) $\frac{1}{4}\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & -1 \\ 1 & -3 & -1 \end{pmatrix}$; (b) 5; (c) $b = a$ or $c = a$ or $c = b$.

15. (b) $\frac{\pi^4}{90}$; (c) $A_0 = 1; A_n = 2(-1)^n\cos n\pi x/(n^2\pi^2+1)$.

16. (c) $\underline{\boldsymbol{E}} = \frac{r}{r^3}$; $\boldsymbol{\nabla} \times \underline{\boldsymbol{E}} = \underline{\boldsymbol{0}}$.

17. (c) (i) $P(A \cap B \cap C)$; (ii) $P(A)$; (iii) $P(B)$; (iv) $P(A \cup B)$; (v) $P(B|A)$; (d) $N = 3744$.

18. (a) $e^2 - 3$; (b) $\frac{1}{6}$; (c) $\pi - \frac{2}{3}$.

19. (a) zero, $\frac{\pi}{4}$; (b) zero.