A Tutorial on Multiple Integrals
(for Natural Sciences / Computer Sciences Tripos Part IA Maths)

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This tutorial gives some brief examples of where multiple integrals arise in science, and then tackles the issue which causes most confusion for students: how to work out the limits on the integrals when integrating over a given region.

1. Some Examples

(a) The multiple integral $\iiint_R dx\, dy\, dz$ over some region of three dimensional space *R* represents the *volume* of the region *R*. In this example, the integrand (the function we are integrating) is equal to one.

(b) If the integrand is not simply one, then we might get something like $\iiint_R \rho(x, y, z)\, dx\, dy\, dz$. If $\rho(x, y, z)$ is the mass density at any point within the region *R*, then the multiple integral represents the *total mass* within the region *R*. In effect, one is multiplying density by volume to get the mass, but allowing for the fact that the density might vary over the region. If $\rho(x, y, z)$ is the charge density rather than the mass density, then the multiple integral represents the *total charge* contained within the region.

(c) The multiple integral $\iint_R z(x, y)\, dx\, dy$ can be regarded as the *volume* under a surface $z(x, y)$ within a two dimensional region of the *xy*-plane *R*.

2. How to Find the Limits

The process that students most struggle with at first is how to find the limits on the integrals, given a region. There exists a systematic method for doing this, and if you follow it, you should get the correct answer; if you do not follow it, you may well not get the correct answer. We will take a specific example to demonstrate. Imagine we wish to find the volume under the surface $z = x + y$ within the region shown below:

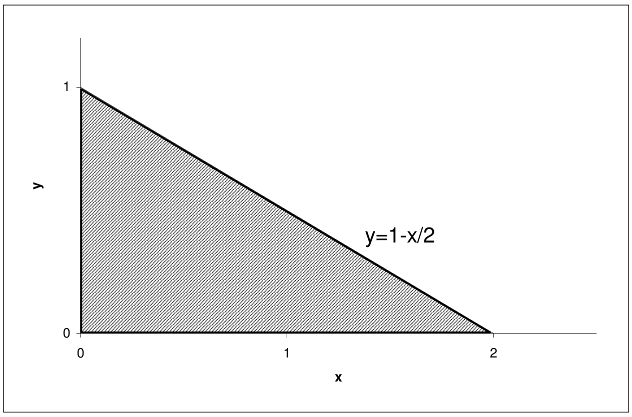
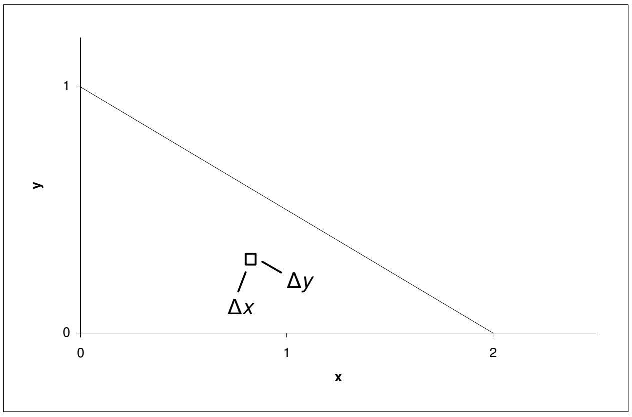


Figure 1

We can imagine making up the volume from a set of volume elements, each of height $z = x + y$ and base dimensions Δx and Δy. Viewed from above, one element would look like this:



and from the side:

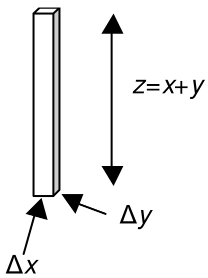


Figure 2

The volume would then be the sum of the volumes of the elements needed to cover the entire region:

$$\sum_y \sum_x (x + y)\,\Delta x\,\Delta y$$

We can carry out both summations using integration, by allowing Δx and Δy to tend to zero. But we have to be careful of the limits on the integrals. Let's assume we choose to do the *x*-sum (ie *x*-integral) first. We'll also do it *y*-first later on.

Imagine that we replicate the element so that we make a line of elements, stretching in the *x*-direction:

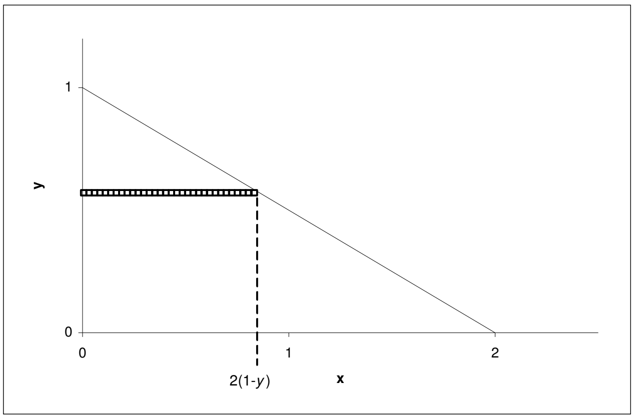


Figure 3

Ask yourself: what is the minimum *x* of these elements, and what is the maximum *x* of these elements? The minimum is 0 and the maximum is $2(1-y)$. Note that the maximum is *not* 2: it depends on where the line of elements is on the *y*-axis. This procedure gives us the limits for the integral in *x*.

To find the *y*-limits, we now imagine our single line of elements above being replicated to cover the entire region:

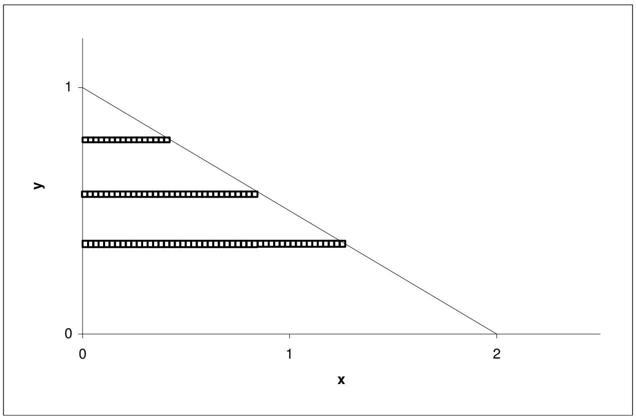


Figure 4

Ask yourself: what is the minimum *y* of these lines of elements, and what is the maximum *y* of these lines of elements? The minimum is 0 and the maximum is 1. This gives us the limits for the integral in y. Hence the overall integral is:

$$\int_{y=0}^{y=1} \int_{x=0}^{x=2(1-y)} (x+y)\, dx\, dy$$

When evaluating the integral, we do the *x*-integral first. When reading an expression like the one above, do not be fooled into thinking that the *y*-integral is done first because its integral sign comes first. We do the inner integral first, and then the outer integral. While we are doing the x-integral, we regard y as a constant, which is consistent with Figure 3. So we get:

$$\int_{y=0}^{y=1} \left[x^2/2 + xy\right]_{x=0}^{x=2(1-y)} dy$$

$$= \int_{y=0}^{y=1} \left[2(1-y)^2 + 2(1-y)y\right] dy$$

$$= \int_{y=0}^{y=1} (2-2y)\, dy \;=\; \left[2y - y^2\right]_0^1 \;=\; 1$$

If instead we chose to do the *y*-sum (ie the *y*-integral) first, then the procedure for finding the limits consists of imagining a line of elements along *y* for some fixed *x*:

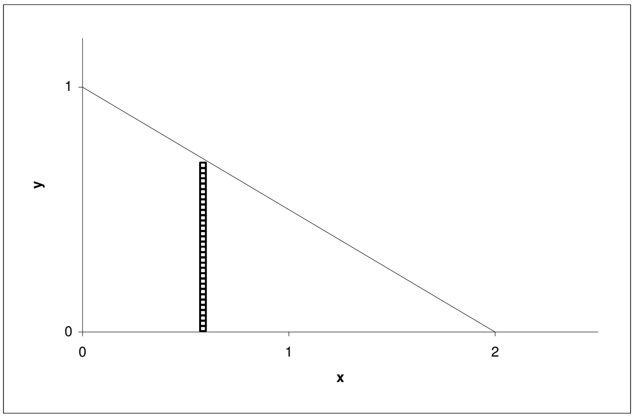


Figure 5

That would give us a minimum *y* of 0 and a maximum *y* of $1 - x/2$. We would then replicate the lines of elements to make it cover the region:

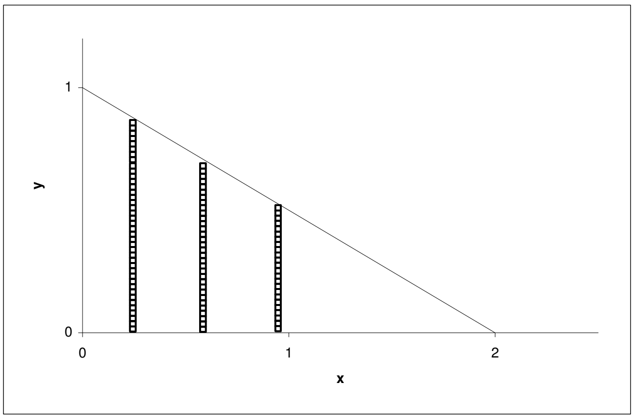


Figure 6

The minimum *x* is 0 and the maximum *x* is 2. So the integral is:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=1-x/2} (x+y)\, dy\, dx$$

$$= \int_{x=0}^{x=2} \left[xy + y^2/2 \right]_{y=0}^{y=1-x/2} dx$$

$$= \int_{x=0}^{x=2} \left[x(1-x/2) + (1-x/2)^2/2 \right] dx$$

$$= \int_{x=0}^{x=2} \left[-\frac{3}{8}x^2 + \frac{x}{2} + \frac{1}{2} \right] dx = \left[-\frac{x^3}{8} + \frac{x^2}{4} + \frac{x}{2} \right]_0^2 = 1.$$

The two routes (*x*-first and *y*-first) must give the same answer. Sometimes one is quicker than the other, and occasionally one is simply not feasible when the other is, so you may need to choose the order of integration with care.

3. Multiple Integrals in non-Cartesian Coordinate Systems

Although the examples above all use the Cartesian coordinate system, it is common to switch to a different coordinate system to do multiple integrals. The reason for this is that the integrals may be difficult in Cartesians. For example, if one wishes to integrate $x^2 + y^2$ over a unit circle, following the procedure above, the limits would be $x = -\sqrt{1-y^2}$ to $x = +\sqrt{1-y^2}$ and then y = -1 to +1. After substituting in the *x*-limits, one is left with an unpleasant integral in *y*. By contrast, if one does the integral in plane polars, then the limits are much simpler. One can still use the conceptual process described above. Firstly, imagine a single plane polar element within the circle:

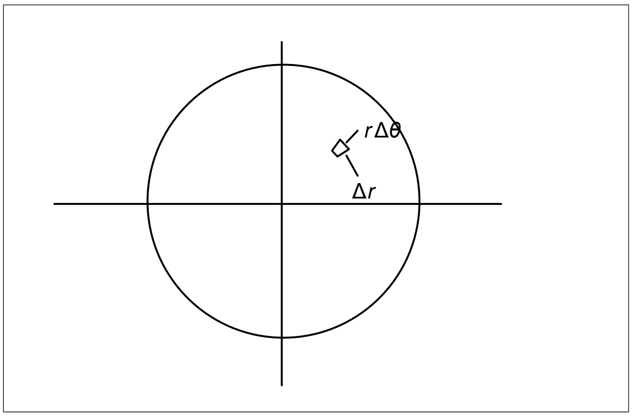


Figure 7

Then replicate the element, firstly in the *r*-dimension to produce a line of elements, and then in the *θ*-dimension to cover the whole circle. The *r*-limits are simply 0 to 1 and the θ-limits are 0 to 2π. We can replace the integrand $x^2 + y^2$ with r^2. But what do we replace *dx dy* with? A rigorous answer to this question turns out to be complicated, but a simple procedure will give us the answer. In Figure 7, one can see that the sides of the element are of length Δr and $r\Delta\theta$, and if Δr and $\Delta\theta$ are very small then the element is approximately rectangular. So its area is $\Delta r\, r\Delta\theta$, or $r\Delta r\Delta\theta$. It turns out that if we simply replace *dx dy* with $rdrd\theta$, we get the correct integral:

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r^2 \; rdr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r^3 \, dr \, d\theta$$

At this point we could do the r integral, followed by the θ integral, or we can speed things up a little by noticing that the limits are constants. This allows us to split the double integral into a product of one dimensional integrals, and process them in parallel:

$$\left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \left(\int_{r=0}^{r=1} r^3 \, dr \right)$$

$$= [\theta]_0^{2\pi} \left[\frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}$$