Formulae Sheet for Economics Part IIA Paper 6

You should be able to recall these formulae in the exam, in whatever form you prefer. I cannot guarantee that the list is exhaustive, but it should cover the great majority of the formulae you will need.

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Original version: 24 March 2006; last updated: 13 December 2012

Calculus

$$\frac{d}{dx}\ln(kx) = \frac{1}{x}$$

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

Differentiation of a product: $\dfrac{d(uv)}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

Integration by parts: $\int u\dfrac{dv}{dx}dx = [uv] - \int v\dfrac{du}{dx}dx$

Quadratic formula: $ax^2 + bx + c = 0$ has solution $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Definition of a convex function: $f((1-t)a + tb) \le (1-t)f(a) + tf(b)$ for $0 \le t \le 1$

Total differential: $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

$\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ unless *f* is discontinuous

Implicit function theorem: along an isoquant of *f*: $\dfrac{dy}{dx} = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

Homogeneous function $f(x_1, x_2, \ldots, x_n)$, of degree *m* obeys:

$f(tx_1, tx_2, \ldots, tx_n) = t^m f(x_1, x_2, \ldots, x_n)$ **and** (Euler) $x_1\dfrac{\partial f}{\partial x_1} + x_2\dfrac{\partial f}{\partial x_2} + \ldots + x_n\dfrac{\partial f}{\partial x_n} = mf$

Binomial series: $(1+x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \dfrac{n(n-1)(n-2)x^3}{3!} + \ldots$

or: $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots + nab^{n-1} + b^n$

Maclaurin series: $f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \ldots$

Taylor series in 1D: $f(a+x) = f(a) + xf'(a) + \frac{x^2 f''(a)}{2!} + \frac{x^3 f'''(a)}{3!} + \ldots$

exponential series: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

log series: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

Taylor series in 2D:

$$f(a+x,b+y) = f(a,b) + \left(x\left.\frac{\partial f}{\partial x}\right|_{(a,b)} + y\left.\frac{\partial f}{\partial y}\right|_{(a,b)} \right) + \frac{1}{2!}\left(x^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)} + 2xy\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x,y)} + y^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(a,b)} \right) + \ldots$$

ie:

$$f(a+x,b+y) = f(a,b) + (x,y)\left.\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}\right|_{(a,b)} + \frac{1}{2!}(x,y)\left.\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\right|_{(a,b)} \begin{pmatrix} x \\ y \end{pmatrix} + \ldots$$

l'Hôpital: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ when $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) =$ *both* 0 or *both* $\pm\infty$

$\lim_{n \to \infty} (1 + x/n)^n = e^x$

Geometric progression: $\sum_{i=0}^{\infty} ar = \frac{a}{1-r}$

& you should either know, or be able to work out from this, that $\sum_{i=0}^{n} ar = \frac{a(1-r^{n+1})}{1-r}$

In whatever form you choose to remember them: Mean Value Theorem, Envelope Theorem, Shephard's Lemma, Roy's Identity.

Statistics

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$

Bayes: $P(A \mid B) = \dfrac{P(B \mid A)P(A)}{P(B)}$

$P(A \cap B) = P(A)P(B)$ if A and B are independent

$P(A \cap B) + P(A \cap \overline{B}) = P(A)$

Binomial probability of r successes n trials = ${}^{n}C_{r}p^{r}(1-p)^{n-r}$ where p = probability of success in any one trial.

Poisson probability of r successes = $\dfrac{\lambda^{r}e^{-\lambda}}{r!}$

Normal distribution is $N(\mu,\sigma^{2}) = \dfrac{1}{\sqrt{2\pi\sigma^{2}}}e^{\frac{-(x-\mu)^{2}}{2\sigma^{2}}}$

Beta distribution $Beta(\alpha,\beta) = Kx^{\alpha-1}(1-x)^{\beta-1}$ (K=normalising constant) for $0 \le x \le 1$

Conditional density function: $g(y \mid x) = \dfrac{f(x,y)}{h(x)}$ where f(x,y) = joint density function

Marginal density function: $h(x) = \int f(x,y)dy$

Continuous form of Bayes' Theorem: $f(x \mid data) = \dfrac{prob(data \mid x)f(x)}{\int prob(data \mid x)f(x)dx}$

Uncentred and centred r^{th} moments: $\mu_{r}^{'} = E(x^{r})$ and $\mu_{r} = E((x-\mu)^{r})$

$E(x^{r}) = \sum_{x} x^{r}\,\Pr(x)$ (discrete) or $E(x^{r}) = \int x^{r}f(x)dx$ (continuous)

$E(X_1 + X_2) = E(X_1) + E(X_2)$

$E(X_1 + k) = E(X_1) + k$

$E(kX_1) = kE(X_1)$

$E(X_1X_2) = E(X_1)E(X_2)$ <u>if</u> X_1 and X_2 are independent

$\text{var}(X) = E[(X - E(X))^{2}] = E(X^{2}) - [E(X)]^{2}$

$\text{var}(X_1 + k) = \text{var}(X_1)$

$\text{var}(kX_1) = k^2\ \text{var}(X_1)$

$\text{var}(X_1 \pm X_2) = \text{var}(X_1) + \text{var}(X_2)$ if X_1 and X_2 are independent

$\text{var}(X_1 \pm X_2) = \text{var}(X_1) + \text{var}(X_2) \pm 2\,\text{cov}\,\text{ar}(X_1, X_2)$ more generally

Moment generating function: $M_X(t) = E(e^{tx})$ and $\left.\dfrac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu_r'$

$M_{X_1+X_2+X_{31}+\ldots}(t) = M_{X_1}(t) M_{X_2}(t) M_{X_3}(t)\ldots$

$M_{X+a}(t) = e^{at} M_X(t)$

$M_{bX}(t) = M_X(bt)$

Moment generating function of N(0,1) is $e^{t^2/2}$

Cramer-Rao lower bound: $\text{var}(\widehat{\theta}) \geq \left[-E\left(\dfrac{d^2 \log(L)}{d\theta^2}\right)\right]^{-1}$

Chebyshev: $\Pr(|X - \mu| \geq \varepsilon) \leq \dfrac{\sigma^2}{\varepsilon^2}$

Linear Algebra

$A + (B + C) = (A + B) + C$

$A(B + C) = AB + AC$

$(A + B)^T = A^T + B^T$

$AB \neq BA$, in general

$A^{-1}A = AA^{-1} = I$

$A(BC) = (AB)C$

$(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1}A^{-1}$

$\det(A^T) = \det(A)$

$\det(AB) = \det(A)\det(B)$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

Rules of determinants:
Can subtract a multiple of any row from any other row without affecting value
Can expand along any row
Swapping two rows reverses the sign of the determinant
Multiplying all terms in a single row by λ multiplies the determinant by λ
Any rule that applies to rows also applies to columns

Formula for the inverse of a matrix, using cofactors: $A^{-1} = \dfrac{adj(A)}{\det(A)}$ and hence A^{-1} exists if and only if $\det(A) \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Cramer's Rule (in whatever form you wish to remember it)

Linear independence: $\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \ldots + \alpha_n \underline{x}_n = \mathbf{0}$ if and only if $\alpha_i = 0$ for all i

Eigenvalues and eigenvectors: $A\underline{v} = r\underline{v}$ and $\det(A - rI) = 0$

Sum of eigenvalues of A = Trace of A

Product of eigenvalues of A = Determinant of A

Diagonal form $D = P^{-1}AP$ exists if P^{-1} exists, ie if the eigenvalues of A are *distinct*.

The two key rules of vector spaces: if $\underline{u}$ and $\underline{v}$ lie in the space, then $\underline{u} + \underline{v}$ and $r\underline{u}$ must also lie in the space.

Fundamental Theorem of Linear Algebra
Dim(Null(A)) = No. of columns in A - Rank(A)