Input-Output Analysis (A Summary)

Input-Output Analysis is used to model an economy where one has a number of sectors or goods, where the output from one sector may be needed as an input for another sector. Let's take a simple example (based on Emanuela Sciubba's examples sheet for Economics Part IIA Paper 6 from 2002-3):

In an economy there are three goods: steel, tractors and corn. One unit of steel requires an input of 0.3 units of steel, 0.1 tractors and 0.1 units of corn; one tractor requires an input of 0.6 units of steel, 0.2 tractors and 0.1 units of corn; one unit of corn requires 0.3 tractors and 0.6 units of corn.

We can express this mathematically as follows. Let the total units of steel, tractors and corn we make be (x_s, x_t, x_c). Some of this "gross output" of steel is used to make steel, tractors and corn; the rest is available for other uses, which we refer to as "final demand", or "net output". Let the final demand for steel, tractors and corn be (d_s, d_t, d_c). Then, considering a balance equation for steel:

gross steel output = (steel we use in making steel, tractors, corn) + net steel output

ie $x_s = 0.3x_s + 0.6x_t + 0x_c + d_s$

and similarly for tractors and corn:

$$x_t = 0.1x_s + 0.2x_t + 0.3x_c + d_t$$
$$x_c = 0.1x_s + 0.1x_t + 0.6x_c + d_s$$

Hence: $$\begin{pmatrix} x_s \\ x_t \\ x_c \end{pmatrix} = \begin{pmatrix} 0.3 & 0.6 & 0 \\ 0.1 & 0.2 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} x_s \\ x_t \\ x_c \end{pmatrix} + \begin{pmatrix} d_s \\ d_t \\ d_c \end{pmatrix}$$

or: $\mathbf{x} = A\mathbf{x} + \mathbf{d}$ .... [1]

where:

$\mathbf{x} = \begin{pmatrix} x_s \\ x_t \\ x_c \end{pmatrix}$ vector of gross output

$A = \begin{pmatrix} 0.3 & 0.6 & 0 \\ 0.1 & 0.2 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}$ input - output matrix

$\mathbf{d} = \begin{pmatrix} d_s \\ d_t \\ d_c \end{pmatrix}$ vector of final demands or net output

Equation [1] is the key equation of input-output analysis.

The questions one usually asks for this kind of economy are:

* "is it possible to find a non-negative solution for $\mathbf{x}$ for *some particular* $\mathbf{d}$?"
* "is it possible to find a non-negative solution for $\mathbf{x}$ for *any* $\mathbf{d}$?"

We are interested in non-negative solutions since these correspond to feasible solutions in the real world. We can attempt to answer these questions by rearranging equation [1]:

$(I - A)\mathbf{x} = \mathbf{d}$ .... [2]

$\mathbf{x} = (I - A)^{-1}\mathbf{d}$ .... [3]

Hence one could test quickly to see if it is possible to find a non-negative solution for $\mathbf{x}$ for *some particular* $\mathbf{d}$ by checking whether $(I - A)^{-1}$ exists and if so, if [3] gives a value for $\mathbf{x}$ that has no negative elements.

What about the more general question of whether it is possible to find a non-negative solution for $\mathbf{x}$ for *any* $\mathbf{d}$? The answer to this is yes, as long as $(I - A)^{-1}$ exists and contains *no negative elements*. If so, A is known as **productive**. This now becomes the key issue. It turns out that there are three ways to test whether A is productive:

(a) if A contains no negative elements and its *columns* sum to less than one, A is productive;
(b) otherwise, if A contains no negative elements and its *rows* sum to less than one, A is productive;
(c) otherwise, you just have to work out $(I - A)^{-1}$ and check that it has no negative elements, to conclude that A is productive.

The proof of (a) and (b) are non-trivial (Simon and Blume have proofs, I think), but you can produce rough justifications fairly easily:

(a) if the columns sum to less than one, then to produce one unit of output in each sector needs less than a total of one unit of inputs
(b) if the rows sum to less than one, then we can produce one unit of output in every sector using less than one unit of input from sector1, sector2....

In our steel, tractors, corn example, rule (a) is sufficient to prove that A is productive. Hence we can be sure that we can meet any final demand by appropriate gross output levels. Using Excel, one finds that

$$(I - A)^{-1} = \begin{pmatrix} 1.801242236 & 1.490683 & 1.118012 \\ 0.434782609 & 1.73913 & 1.304348 \\ 0.559006211 & 0.807453 & 3.10559 \end{pmatrix}$$

which does indeed contain no negative elements.

That is the basic input-output model. One other question which sometimes arises is how do we model labour needs in such a system? Let's say the labour wage rate is w and that the labour requirements to produce one unit of each good are:

$$\mathbf{l} = \begin{pmatrix} l_s \\ l_t \\ l_c \end{pmatrix} .$$

Let's also say that the prices for each good are:

$$\mathbf{p} = \begin{pmatrix} p_s \\ p_t \\ p_c \end{pmatrix} .$$

Then our net income is $\mathbf{p}^T \mathbf{d}$.
Our net expenditure is $w\mathbf{l}^T \mathbf{x}$

So overall, if no profit is made then:

$$\mathbf{p}^T \mathbf{d} = w\mathbf{l}^T \mathbf{x}$$

but by [2]: $\mathbf{d} = (I - A)\mathbf{x}$

so $\mathbf{p}^T (I - A)\mathbf{x} = w\mathbf{l}^T \mathbf{x}$

so $(\mathbf{p}^T (I - A) - w\mathbf{l}^T)\mathbf{x} = 0$

This must be true for *all* $\mathbf{x}$, so $\mathbf{p}^T (I - A) - w\mathbf{l}^T = \mathbf{0}^T$, or $(I - A)^T \mathbf{p} - w\mathbf{l} = \mathbf{0}$.

Hence the vector of prices can be found from $\mathbf{p} = w((I - A)^T)^{-1}\mathbf{l}$.

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