Proof of the Envelope Theorem for Constrained Optimisation

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This document sets out a proof of the Envelope Theorem in the case of constrained optimisation with equality constraints. It is an expanded version of Charles Roddie's proof, given in lectures.

Say we seek to maximise

$$f(x_1, x_2, \ldots, x_n, \alpha) \qquad [1]$$

subject to

$$g^j(x_1, x_2, \ldots, x_n, \alpha) = c^j \quad j = 1 \ldots m \qquad [2]$$

by choosing the x_i. Here, α is a parameter.

Denote the optimal value of *f* by f^* and the optimal value of x_i by x_i^*.

Let the "value function" be defined by

$$v(\alpha) = f^*(x_1^*, x_2^*, \ldots, x_n^*, \alpha) \qquad [3]$$

We seek to find the value of $\frac{dv}{d\alpha}$. Well, differentiating [3] gives

$$\frac{dv}{d\alpha} = \frac{df^*(x_1^*, x_2^*, \ldots, x_n^*, \alpha)}{d\alpha} \qquad [4]$$

Using the chain rule

$$\frac{dv}{d\alpha} = \frac{\partial f^*}{\partial \alpha} + \sum_i \frac{\partial f^*}{\partial x_i^*} \frac{dx_i^*}{d\alpha} \qquad [5]$$

But we know that, at the optimium

$$\frac{\partial L}{\partial x_i} = 0 \text{ for all } i \qquad [6]$$

where *L* is the Lagrangian $L = f - \sum_j \lambda_j (g^j - c^j)$. So

$$\frac{\partial f^*}{\partial x_i^*} - \sum_j \lambda_j \frac{\partial g^j}{\partial x_i^*} = 0 \text{ for all } i \qquad [7]$$

Putting this into [5]

$$\frac{dv}{d\alpha} = \frac{\partial f^*}{\partial \alpha} + \sum_i \sum_j \lambda_j \frac{\partial g^j}{\partial x_i^*} \frac{dx_i^*}{d\alpha} \qquad [8]$$

The order of the sums does not matter, so we can also write this as

$$\frac{dv}{d\alpha} = \frac{\partial f^*}{\partial \alpha} + \sum_j \lambda_j \sum_i \frac{\partial g^j}{\partial x_i^*} \frac{dx_i^*}{d\alpha} \qquad [9]$$

But we also know that at the optimum, the constraint is satisfied, so

$$g^j\left(x_1^*, x_2^*, \ldots, x_n^*, \alpha\right) = c^j \quad j = 1 \ldots m \qquad [10]$$

And differentiating this with respect to α gives:

$$\frac{dg^j}{d\alpha} = \frac{\partial g^j}{\partial \alpha} + \sum_i \frac{\partial g^j}{\partial x_i^*} \frac{dx_i^*}{d\alpha} = 0 \quad j = 1 \ldots m \qquad [11]$$

Hence from [9]

$$\frac{dv}{d\alpha} = \frac{\partial f^*}{\partial \alpha} - \sum_j \lambda_j \frac{\partial g^j}{\partial \alpha} \qquad [12]$$

So

$$\frac{dv}{d\alpha} = \frac{\partial\left(f^* - \sum_j \lambda_j g^j\right)}{\partial \alpha} \qquad [13]$$

And hence

$$\frac{dv}{d\alpha} = \frac{\partial L^*}{\partial \alpha} \qquad [14]$$