A Tutorial on Simple First Order Linear Difference Equations
(for Economics Part I Paper 3)

Corrections to Dr Ian Rudy (http://people.pwf.cam.ac.uk/iar1/contact.html) please.

An example of a simple first order linear difference equation is:

\[ x_t + 2x_{t-1} = 1800 \]  \[1\]

The equation relates the value of \( x \) at time \( t \) to the value at time \((t-1)\). Difference equations regard time as a discrete quantity, and are useful when data are supplied to us at discrete time intervals. Examples include unemployment or inflation data, which are published one a month or once a year. Difference equations are similar to differential equations, but the latter regard time as a continuous quantity. Equation [1] is known as a first order equation in that the maximum difference in time between the \( x \) terms (\( x_t \) and \( x_{t-1} \)) is one unit. Second order equations involve \( x_t \), \( x_{t-1} \) and \( x_{t-2} \).

Equation [1] is known as linear, in that there are no powers of \( x \) beyond the first power.

There are various ways of solving difference equations. In lectures, you may simply be given a formula for the solution for a general difference equation. This is fine if you have a good memory, but is not terribly interesting. Another method begins from the assumption that we know \( x_0 \), and can then use [1] to find the value of \( x_1 \). Having done this, we can then use [1] again to find the value of \( x_2 \), and so on. This method is very general in principle, but in practice its usefulness depends on whether we are able to sum the series that appear to get a general expression for \( x_t \).

We will look at a third method of solving [1] in some detail. It is a two-stage process. We first of all look for any solution - no matter how simple it is, or whether it is the complete solution to the equation. When the right hand side of the equation is a constant, as it is in [1], this is quite simple: we just seek a solution:

\[ x_t = x_{t-1} = x^* \]  \[2\]

This is often known as a steady-state or equilibrium solution. For equation [1], we get:

\[ x^* + 2x^* = 1800 \]

so

\[ x^* = 600 \]  \[3\]

In stage two of the process, we look for a more sophisticated solution, such as:

\[ x_t = x^* + z_t \]  \[4\]

In our case, \( x^* = 600 \), and by substituting [4] into [1], we get:

\[ (600 + z_t) + 2(600 + z_{t-1}) = 1800 \]
so \[ z_t + 2z_{t-1} = 0 \] \[ [5] \]

It should be apparent that \([5]\) will always be \([1]\) with zero on the right hand side, and once you realise this, you can save time by jumping straight to \([5]\) from \([1]\).

Equation \([5]\) can be solved in various ways. One way, which very usefully extends to second order equations, is to propose a trial solution of:

\[ z_t = A(\lambda)^t \] \[ [6] \]

by substituting this into \([5]\), one finds:

\[ A(\lambda)^t + 2A(\lambda)^{t-1} = 0 \]

so, cancelling a factor \((\lambda)^{t-1}\):

\[ \lambda + 2 = 0 \] \[ [7] \]

so \[ \lambda = -2 \]

Hence from \([6]\), the solution is:

\[ z_t = A(-2)^t \]

In the case of a second order equation, \([7]\) is replaced by a quadratic in \(\lambda\), from which you will get two values of \(\lambda\) (let’s call them \(\lambda_1, \lambda_2\)), and the solution for \(z_t\) is:

\[ z_t = A(\lambda_1)^t + B(\lambda_2)^t \]

But returning to our first order equation \([1]\), by putting together \([4]\), \([3]\) and \([6]\), we find the solution is:

\[ x_t = 600 + A(-2)^t \] \[ [8] \]

To find \(A\), we need some information about \(x_t\) at one value of \(t\). Most commonly, we will know, or be given information about, \(x_0\), known as an initial condition. For example, if \(x_0 = 601\), then from \([8]\), \(A = 1\), and so:

\[ x_t = 600 + (-2)^t \] \[ [9] \]

In summary, the solution to difference equations of the form of \([1]\) is:

\[ x_t = x^* + z_t \]
where \( x^* \) is the steady state solution and \( z_t \) is found by putting zero on the right hand side of the difference equation, replacing \( x_t \) by \( z_t \) and using a trial solution of \( z_t = A(\lambda)^t \) to find \( \lambda \). Hence the general solution is:

\[
x_t = x^* + A(\lambda)^t
\]  

[10]

The value of the constant \( A \) can be found from the initial condition(s).

You will come across some other terminology in books: \( x^* \) is also known as the particular solution or particular integral, and \( z_t \) is known as the complementary solution or complementary function. We then have:

general solution = particular solution + complementary solution.

Having found the solution to [1], a question which often arises is how \( x_t \) varies with \( t \). By plotting [10] against time, you should be able to see that there are four situations we might encounter:

\( \lambda < -1 \): unstable, oscillating

\( -1 < \lambda < 0 \): stable, oscillating

\( 0 < \lambda < 1 \): stable, not oscillating

\( \lambda > 1 \): unstable, not oscillating