A Tutorial on Simple First Order Linear Difference Equations
(for Economics Part I Paper 3)

Corrections to Dr Ian Rudy (http://people.pwf.cam.ac.uk/iar1/contact.html) please.

An example of a simple first order linear difference equation is:

$$x_t + 2x_{t-1} = 1800 \qquad [1]$$

The equation relates the value of x at time t to the value at time (t-1). Difference equations regard time as a discrete quantity, and are useful when data are supplied to us at discrete time intervals. Examples include unemployment or inflation data, which are published one a month or once a year. Difference equations are similar to differential equations, but the latter regard time as a continuous quantity. Equation [1] is known as a first order equation in that the maximum difference in time between the x terms (x_t and x_{t-1}) is one unit. Second order equations involve x_t, x_{t-1} and x_{t-2}. Equation [1] is known as linear, in that there are no powers of x_t beyond the first power.

There are various ways of solving difference equations. In lectures, you may simply be given a formula for the solution for a general difference equation. This is fine if you have a good memory, but is not terribly interesting. Another method begins from the assumption that we know x_0, and can then use [1] to find the value of x_1. Having done this, we can then use [1] again to find the value of x_2, and so on. This method is very general in principle, but in practice its usefulness depends on whether we are able to sum the series that appear to get a general expression for x_t.

We will look at a third method of solving [1] in some detail. It is a two-stage process. We first of all look for *any* solution - no matter how simple it is, or whether it is the complete solution to the equation. When the right hand side of the equation is a constant, as it is in [1], this is quite simple: we just seek a solution:

$$x_t = x_{t-1} = x^* \qquad [2]$$

This is often known as a steady-state or equilibrium solution. For equation [1], we get:

$$x^* + 2x^* = 1800$$

so

$$x^* = 600 \qquad [3]$$

In stage two of the process, we look for a more sophisticated solution, such as:

$$x_t = x^* + z_t \qquad [4]$$

In our case, $x^* = 600$, and by substituting [4] into [1], we get:

$$(600 + z_t) + 2(600 + z_{t-1}) = 1800$$

so $$z_t + 2z_{t-1} = 0 \qquad [5]$$

It should be apparent that [5] will always be [1] with zero on the right hand side, and once you realise this, you can save time by jumping straight to [5] from [1].

Equation [5] can be solved in various ways. One way, which very usefully extends to second order equations, is to propose a trial solution of:

$$z_t = A(\lambda)^t \qquad [6]$$

by substituting this into [5], one finds:

$$A(\lambda)^t + 2A(\lambda)^{t-1} = 0$$

so, cancelling a factor $A(\lambda)^{t-1}$: $$\lambda + 2 = 0 \qquad [7]$$

so $$\lambda = -2$$

Hence from [6], the solution is:

$$z_t = A(-2)^t$$

In the case of a second order equation, [7] is replaced by a quadratic in λ, from which you will get *two* values of λ (let's call them λ_1, λ_2), and the solution for z_t is:

$$z_t = A(\lambda_1)^t + B(\lambda_2)^t$$

But returning to our first order equation [1], by putting together [4], [3] and [6], we find the solution is:

$$x_t = 600 + A(-2)^t \qquad [8]$$

To find A, we need some information about x_t at one value of *t*. Most commonly, we will know, or be given information about, x_0, known as an initial condition. For example, if $x_0 = 601$, then from [8], $A = 1$, and so:

$$x_t = 600 + (-2)^t \qquad [9]$$

In summary, the solution to difference equations of the form of [1] is:

$$x_t = x^* + z_t$$

where x^* is the steady state solution and z_t is found by putting zero on the right hand side of the difference equation, replacing x_t by z_t and using a trial solution of $z_t = A(\lambda)^t$ to find λ. Hence the general solution is:

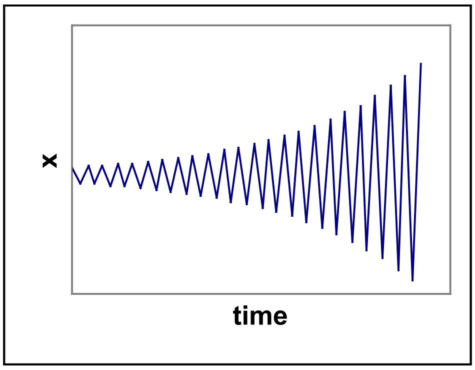
$$x_t = x^* + A(\lambda)^t \qquad [10]$$

The value of the constant A can be found from the initial condition(s).

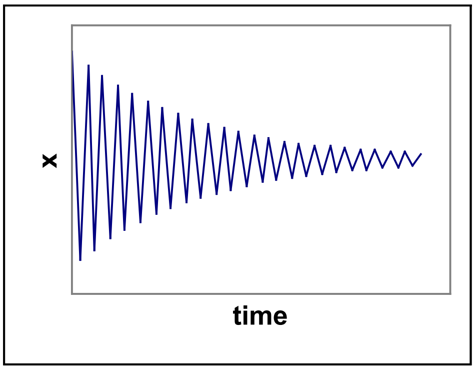
You will come across some other terminology in books: x^* is also known as the particular solution or particular integral, and z_t is known as the complementary solution or complementary function. We then have:

general solution = particular solution + complementary solution.

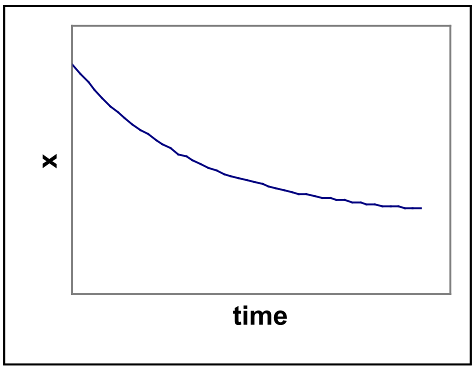
Having found the solution to [1], a question which often arises is how x_t varies with *t*. By plotting [10] against time, you should be able to see that there are four situations we might encounter:



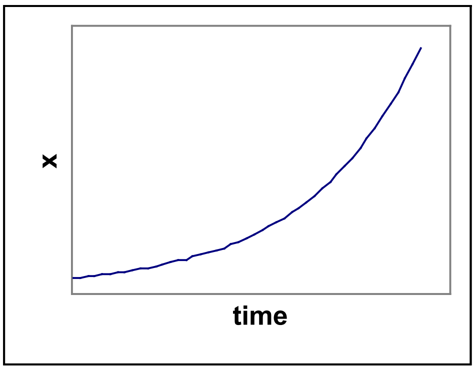
$\lambda < -1$: unstable, oscillating



$-1 < \lambda < 0$: stable, oscillating



$0 < \lambda < 1$: stable, not oscillating



$\lambda > 1$: unstable, not oscillating