RECORD BREAKING AND TEMPORAL CLUSTERING∗

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Abstract. This paper presents a model of learning under limited awareness, in which agents gradually discover what is achievable. A record breaking performance spurs the agents to try harder and thus temporarily increases the probability of new records. Record breaking trails off when the record approaches the true limit to performance. Under naive updating, agents take the current record as the upper bound and effort is increasing over time. This model is contrasted with a standard Bayesian setting. It is shown that effort under Bayesian updating may be non-monotone but is higher than under naive updating, ceteris paribus.

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“Though physiology may indicate respiratory and circulatory limits to muscular effort, psychological and other factors beyond the ken of physiology set the razor’s edge of defeat or victory and determine how close an athlete approaches the absolute limits to performance” - Roger Bannister

1. Introduction
On May 6, 1954, 25 year-old Roger Bannister was the first in history to run one mile in less than four minutes. With a time of 3 minutes and 59.4 seconds, he had accomplished an astonishing feat, something his contemporaries deemed impossible and which is routinely compared to Edmund Hillary and Tenzing Norgay’s conquest of Mount Everest almost exactly a year earlier on May 29, 1953. According to Myers (2002), “for decades it was considered beyond human capacity, virtually in physiological principle, to run a mile inside four minutes”. As it turned out, there was no real barrier of four minutes, only a perceived one. But perceptions matter.

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1Quoted in Myers (2002).
Before Bannister’s achievement, the fastest time remained unchanged for a decade. In contrast, it took a meagre seven weeks for Bannister’s record to be broken by John Landy. Landy said of his thinking before the breaking of the four minute mile: “I honestly felt, certainly after I’d run half a dozen 4.2s, that there was a bit of a barrier there”. In contrast, afterwards he thought that “if he [Bannister] can run as fast as that, so can I”.

The present paper seeks to model this type of unawareness and to study how the observation of hitherto unseen achievements can act to motivate subsequent decision makers and spur them to pursue further achievements. To this end, I present a sequential decision problem in which agents are, in a literal sense, unaware of the true limits to performance. Over time, as outcomes are observed, the agents gradually discover what is actually achievable and adapt their behavior accordingly. Agents value higher outcomes, which are in turn probabilistic functions of costly effort. Agents thus trade off effort costs against expected rewards. The twist is that the expected rewards are influenced by expectations about what is feasible and hence on the way in which expectations are formed and updated.

I consider three main settings. First, I consider what I term naive updating, in which the agent takes the current record as the limit to achievement. This type of updating is also consistent with that of a non-naive but uncertainty averse agent. I show that with naive updating, equilibrium effort is increasing over time and that although the agent learns slowly, he eventually behaves as would a fully Bayesian agent (i.e. learning is complete). Under an increasing frontier of what it is feasible to achieve, the passage of time creates the potential for incremental innovations, while the additional effect of increased effort resulting from new records, creates the potential for more radical innovations, interpreted as outcomes well above past accomplishments. I argue that naive updating creates the potential for temporal clustering of such radical innovations.

Second, I consider two versions of the model with fully Bayesian updating but where the agent behaves myopically. In the simplest setting, the true frontier is fixed and unobserved but drawn from a known distribution. The agent then updates beliefs in a standard way as information arrives. I show that under Bayesian updating, the agent chooses higher effort than under naive updating and that equilibrium effort is not necessarily increasing over time. I then extend the model to one in which the frontier is unknown but also may increase over time. I show that in this setting, the agent chooses higher effort than under an unknown but fixed frontier.

Third, I consider the issue of social optimality and optimal experimentation under forward-looking behavior. I argue that there is a priori no reason to expect that a benevolent social planner would exert higher effort than myopic agents. While a planner would internalize the informational externalities that current effort has on future decisions (and myopic agents do not), experimentation is not necessarily linked to higher effort.

The paper is related to a number of different literatures. First, in a broad sense the present paper is a contribution to the literature on innovation and innovation waves, exemplified by Barzel (1972), Helpman and Trajtenberg (1998) and Andergassen and

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Nardini (2005). At an abstract level, an innovation is an improvement on the existing level of achievement. Similarly, a new record is simply a result which is better than all previous results. Second, the paper is related to the literature on learning by doing and production under demand uncertainty, such as McLennan (1984), Zeira (1987, 1994), Rob (1991) and Jovanovic and Nyarko (1995). Third, the naive updating version of the model is related to papers on non-Bayesian updating and unawareness, such as Muth (1986), Walley (1996), Smithson et al. (2000), Ozbay (2008) and Jovanovic (2009). Fourth, there are similarities between this paper and the literatures on optimal experimentation, sequential sampling and social learning.\(^3\) Optimal experimentation has been analyzed by Easley and Kiefer (1988), Kiefer (1989), Kiefer and Nyarko (1989), Aghion et al. (1991) and Wieland (2000). Last, this paper is related to a very large statistics literature on records times series and estimation.\(^4\)

In Section 2, I set out the decision problem faced by an agent when there is a known limit to achievement. In Section 3, I analyze the problem faced by a naive decision maker (or equivalently, a decision maker who maximizes maxmin expected utility) who takes the current record as the limit to possible achievement. In Section 4, I extend the analysis to a fully rational setting, in which the agent forms standard Bayesian beliefs over the possible feasibility frontiers, when the frontier is fixed (but unknown) at the outset. In Section 5, I extend the Bayesian analysis to a setting with a growing frontier. In Section 6, I discuss the issue of social optimality and optimal experimentation. Section 7 concludes.

2. The Model

Time is discrete and the horizon is infinite. In each period, a new agent appears who lives for just that period.\(^5\) In period \(t = 1, 2, ...,\) an agent exerts effort \(e \geq 0\) to produce some random outcome \(X\) drawn from a continuously differentiable distribution \(G \equiv G(x_t; e, z_t)\) on some interval \((0, z]\) with \(z \leq \infty\), which allows a probability density function \(g\).\(^6\) I will denote a realization of the outcome by \(x\). I explicitly parametrize the distribution function by the upper bound of the support \(z\) because in what follows, \(z\) will be either unknown or allowed to change over time (or both).

There are several possible interpretations of the outcome \(X\). It can be thought of as some tangible quantity such as a physical accomplishment (speed, distance, strength) or profits. Alternatively, \(X\) could stand for some measure of knowledge, with zero being complete ignorance and the frontier \(z\) being complete knowledge.\(^7\)

Given effort \(e\) and outcome \(x\), the agent’s utility is given by the separable function

\[
 u(x) - c(e) \tag{1}
\]

where the utility of outcome \(u\) is bounded, continuous, strictly increasing and concave.

\(^3\)Surveys of these literatures include DeGroot (2004) and Chamley (2004).

\(^4\)See Toxvaerd (2005) for detailed references and for plots of record series for different athletics disciplines.

\(^5\)Equivalently, the agent is infinitely lived but myopic. This distinction will play no role under naive updating.

\(^6\)Equivalently, \(G\) can be defined on \(\mathbb{R}_+\), with \(z \equiv \inf\{x : G(x; e) = 1\}\).

\(^7\)According to Dewar and Dutton (1986), the distinction between an incremental and a radical innovation depends on the “degree of new knowledge embedded in the innovation”.
and the disutility of effort \( c \) is continuous, strictly increasing and convex.\(^8\)

Suppose that the frontier is some known constant \( z > 0 \). Then the agent’s optimization problem is to solve

\[
\max_{e \geq 0} \left\{ \int_0^z u(x) g(x; e, z) dx - c(e) \right\} \tag{2}
\]

The first-order condition for optimal effort \( \hat{e} \), given some frontier \( z \), is

\[
\int_0^z u(x) g_e(x; \hat{e}, z) dx - c_e(\hat{e}) = 0 \tag{3}
\]

This simply states that at the optimum, the expected marginal return to effort is zero. The associated second-order condition for optimality is

\[
\int_0^z u(x) g_{ee}(x; \hat{e}, z) dx - c_{ee}(\hat{e}) < 0 \tag{4}
\]

Note that when the frontier \( z \) is known, then the same optimal effort level will be exerted in each period, regardless of the observed outcome sequence. Thus the outcomes will be independently and identically distributed across time.

Throughout, I maintain the following assumption:

**A1** \( G(x; e, z) \leq G(x; e', z) \) for all \( x \) and \( z \) and \( e \geq e' \).

Assumption A1 simply states that the distribution \( G \) is stochastically increasing in effort (i.e. it shifts in the sense of first-order stochastic dominance). This means that higher effort leads to higher outcomes in expectation.

For use in the Bayesian analysis contained in later sections, I also introduce the following assumption:

**A2** \( g(x; e, z)g(x'; e, z') \geq g(x; e, z')g(x'; e, z) \) for all \( e \geq 0 \), \( x > x' \) and \( z > z' \).

Assumption A2 requires the outcome distribution \( G \) to satisfy the monotone likelihood ratio property in the feasibility frontier \( z \). This assumption will be relevant and useful only in the analysis of the Bayesian versions of the model expounded in Sections 4 and 5.

Next, I make the following assumption:

**A3** \( g(x; e, z) > 0 \) for all \( x \in [0, z] \) and \( e \geq 0 \).

Assumption A3 means that the probability density is strictly positive for all feasible outcomes, i.e. the outcome distribution has full support.

A last assumption is imposed on the problem:

**A4** \( \int_0^z u(x) g_{ez}(x; e, z) dx + u(z) g_e(z; e, z) > 0 \) for all \( z \).

\(^8\)One may rewrite the problem and specify the agent’s utility as \( u(r(x)) \), where \( r(x) \) is the expected rank in some tournament setting and \( r' > 0 \).
Assumption A4 is a sufficient joint condition on \( u \) and \( G \) for optimal effort to be monotone in the perceived frontier, ensuring that the dynamics of the model are interesting. In essence, this assumption ensures that when potential rewards are higher, optimal effort is higher.\(^9\)

### 2.1. Functional Form Examples.

For the purpose of illustration, I next give two functional form examples for the outcome distribution \( G \). The first is based on the uniform distribution. For all \( x \in [0, z] \), let \( X \sim G(x; e, z) \) where the cumulative distribution function is given by

\[
G(x; e, z) = \left( \frac{x}{z} \right)^{\lambda e}
\]

where \( \lambda > 0 \). For \( \lambda e = 1 \), this distribution reduces to the uniform on the interval \([0, z]\).

It is straightforward to verify that this distribution satisfies Assumptions A1 and A3. Also, Assumption A2 is satisfied with equality. In Appendix A, I show that for \( z > 1 \), this choice of \( G \) satisfies Assumption A4 under the additional assumption that the agent is risk neutral, i.e. that \( u(x) = x \).

Another possible choice is the following distribution, which is based on the truncated Pareto distribution:

\[
G(x; e, z) = \left( \frac{1 - x^{-\rho}}{1 - z^{-\rho}} \right)^{\lambda e}
\]

where \( \lambda > 0 \). For \( \lambda e = 1 \), this distribution reduces to the truncated Pareto distribution. It is easily verified that this distribution satisfies Assumptions A1 and A3 and Assumption A2 with equality. Under risk neutrality, it also satisfies Assumption A4 under additional but mild restrictions, including the condition that equilibrium effort is always bounded sufficiently away from zero.

### 3. Naive Updating

In this section, I analyze the decision problem of the agent under what I term *naive updating*. The type of problem that inspired this paper was the kind faced by e.g. athletes trying to break long-standing records. By their very nature, records have never been broken in the past and therefore past experience may not be useful in making probabilistic assessments about their future occurrence. There are some things that have never happened but which no reasonable person would exclude as being *impossible* (rather than having a very low probability of occurring). For example, the simultaneous collision of ten airplanes over the Atlantic is extremely unlikely, but readily recognized as being *possible*, at least in principle. In contrast, there are achievements that are simply not possible under current conditions, say running a mile in 10 seconds. A naive approach to the latter type of uncertainty is to simply assume that the agent is not fully aware of states of nature that have never materialized (and which cannot be deduced by analogy) and therefore disregards these in making

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\(^9\)A similar type of assumption is implicitly maintained in the models of Zeira (1987, 1994) and Rob (1991), in which firms always wish to build enough capacity to meet potential demand, i.e. the costs of doing so do not rise too fast.
There is some experimental evidence that subjects under-estimate or entirely ignore unobserved alternatives when faced with *sample space ignorance* (see e.g. Smithson et al., 2000 and Smithson, 2000).

An alternative and less naive interpretation is that agents, while not unaware of the limits to achievement, are uncertainty averse in the sense of Gilboa and Schmeidler (1989). Paraphrasing them, the agent may have too little information to form a prior belief and therefore considers a set of possible prior beliefs. Such an agent would choose the most pessimistic possible priors and then maximize the resulting expected utility under these priors. Since in the present setup expected utility is increasing in the frontier, any outcomes above the present record are valuable to the agent. In choosing how much probability to assign to such outcomes (which have never materialized in the past), the most pessimistic assessment is to assign probability mass zero to those outcomes, or equivalently, to assign probability zero to the possibility that the true frontier is strictly higher than the present record. But no rational agent can assign positive probability to the event that the true frontier is strictly lower than the present record. This procedure thus effectively assigns probability one to the feasibility frontier and the present record coinciding.

A third alternative interpretation is that agents, while neither unsophisticated nor Bayesian, construct an alternative probability model to deal with the lack of prior information. One such model is the imprecise Dirichlet model due to Walley (1996). According to this model, agents form upper and lower probability bounds for events, based on the frequencies with which they occur. For entirely unobserved events, the lower bound is zero and the upper bound is one, meaning that the agent has vacuous prior probabilities. I other words, when there are events that have never occurred, there is simply not sufficient evidence to act on such events. Agents following this probability model would be behaviorally equivalent to the naive decision maker and to the uncertainty averse decision maker.

In short, I assume that agent $t$ observes all the outcomes of predecessors’ efforts $\{x_1, ..., x_{t-1}\}$ and knows the functional form of the family of distribution functions $\{G(x; e, z)\}_{z \in [0, \infty]}$. Since the agent can infer the effort levels from knowledge of the outcomes, observability of efforts is not an issue of importance. Importantly, the agent does not know (or take into account) that the frontier of feasible outcomes may be expanding, nor its current position $z_t$. Instead, I assume that the agent perceives the outcome to be drawn from the distribution $G(x; e, \pi_t)$, where record $\bar{x}_t$ at time $t = 1, 2, ...$ is the maximum of all outcomes up to time $t-1$, i.e.

$$\pi_t \equiv \max \{x_1, ..., x_{t-1}\} \quad (7)$$

That is, the agent is not aware of (or does not take into account) the possibility of outcomes in the interval $[\pi_t, z_t]$ but instead acts as if $z = \pi_t$ at each point in time $t = 1, 2, ...$

In this setting, upon observing a new record, the agent adapts his distribution

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10. E.g., collisions of two airplanes have occurred in the past, so the agent may reason that it is at least *possible* that a third, fourth and fifth airplane could collide with the other airplanes at exactly the same time, etc.

11. See Walley (1996) for a forceful defense of the appropriateness of the non-Bayesian probability model.
over possible outcomes by shifting probability from lower outcomes towards higher outcomes.\footnote{Ozbay (2008) considers updating under unawareness (i.e. when the agent is unaware of the possible states of nature) and restricts beliefs to those which have full support and which respect initial beliefs. The first restriction means that if the agent assigns positive probability to some state of nature under unawareness, he will continue to do so under awareness. The second restriction means that the conditional distribution over states of which the agent is aware, coincides with the conditional distribution over those states under awareness. These restrictions seem to be stronger than those imposed in the present paper.}

The naive agent’s equilibrium effort at time $t = 1, 2, \ldots$ is characterized by

$$
\int_0^{\bar{x}_t} u(x) g_e(x; \hat{e}, \bar{x}_t) dx - c_e(\hat{e}) = 0
$$

Before considering the case of an increasing feasibility frontier, assume that the maximum achievable outcome is some constant $z \leq \infty$. Because the sequence $\{\bar{x}_t\}_{t=1}^\infty$ is weakly increasing over time, the agent’s effort increases weakly over time. Because of A4, each new record prompts higher effort, thereby pushing perceptions of what is possible outwards. This process continues until the actual frontier $z$ is reached (in case it is finite). This proves the following result:

**Proposition 1.** Under naive updating, equilibrium effort $\hat{e}_t$ is increasing over time.

This result has the following straightforward implication for the probability of record breaking:

**Corollary 2.** As long as $\bar{x}_t < z$, each new record makes it more likely that previous records will be broken and more likely that the new record will itself be broken.

**Proof:** From A1, increased effort increases the outcome distribution in the sense of first-order stochastic dominance. In particular, this implies that for some $\bar{x}_s > \bar{x}_t$,

$$
G(\bar{x}_t; \hat{e}(\bar{x}_t), z) \geq G(\bar{x}_t; \hat{e}(\bar{x}_s), z) \quad (9)
$$

$$
G(\bar{x}_s; \hat{e}(\bar{x}_t), z) \geq G(\bar{x}_s; \hat{e}(\bar{x}_s), z) \quad (10)
$$

where $\hat{e}(\bar{x})$ denotes the optimal choice of effort given the perception that the frontier is given by $\bar{x}$.

In other words, records induce higher effort, which in turn shifts probability mass towards higher outcomes. This increases the probability that both the previous and the new record is subsequently broken. This process ends when the frontier $z$ is reached, i.e. when $\bar{x}_t = z$. Figure 1 illustrates this.

The process can be thought of as discovering by doing, in contrast to learning by doing. In the latter, actions increase the precision of the agent’s information, while in the former, they also let the agent discover new possibilities of which he was previously unaware.

It is clear that the monotonicity of effort will also be present under an increasing frontier (which will be discussed below), because the naive updating procedure is insensitive to how frontiers beyond the current record are located or how they evolve.
Figure 1: Records with a fixed frontier. Records are broken at times \(u, v, \) and \(w\), at which time the frontier \(z\) is reached.

Also, monotonicity of the effort sequence will turn out to be a major difference between the naive and Bayesian settings.

Next, I turn to a setting in which the frontier increases over time. In this setting, there are several interacting effects and therefore it is useful to distinguish between different types of records and innovations. One classification of innovations is incremental versus radical innovations (see e.g. Dewar and Dutton, 1986). Incremental innovations are those with a modest content of new information and are those likely to emerge from small adjustments to processes and from simple repetition, such as those modeled by Jovanovic and Nyarko (1995). In contrast, radical innovations are those that constitute a more significant leap in knowledge and are more likely to be the outcome of intense effort.

Returning to the model, to fix ideas, suppose that the frontier \(z_t\) increases smoothly over time, at a decreasing rate. Specifically, suppose that for exogenous reasons, the feasibility frontier satisfies the following assumption:\(^{13}\)

\[
A5 \quad 0 < z_t < z_{t+1} \text{ for all } t = 1, 2, \ldots \text{ with } z_{t+1} - z_t < z_t - z_{t-1}^{14}.
\]

This assumption reflects the notion that over time, improvements in training regimes, diets and technology increase what it is possible to achieve. I assume that although the frontier strictly increases over time, it does not increase too fast.

Because of A2, the increase in the frontier \(z_t\) causes a gradual outward shift in the outcome distribution, even if effort is maintained at a fixed level. This means that on expectation, records will be broken with some regularity.

In addition to this effect, the agent will have incentives to increase effort each time a record is broken, as described above with a fixed frontier. This means that

\(^{13}\)In the spirit of Barzel (1972), I take the running of time to be a proxy for the evolution of the forces that allow progress to take place.

\(^{14}\)For the continuous time analog of the model, think of the frontier as being concave in time.
with endogenously chosen effort, the probability of broken records, and indeed of radical improvements upon existing records, is higher than what would be the case if innovations were simply the outcome of a gradual expansion of the feasibility frontier. Of course, this argument is true only as long as the records are not too close to the frontier. In other words, radical innovations are only possible, when the discrepancy between perceived and actual limits to performance are sufficiently large.

To complete the treatment of learning under naive updating, I briefly discuss the long run behavior of the learning process. Suppose that the frontier $z$ is fixed and that the agent continues in perpetuity to exert the effort level $e(x_1)$, which is optimal under the initial perceptions. Let $F_{x_n}(x)$ denote the distribution of the record over $n = 1, 2, \ldots$ observations. This distribution is given by

$$F_{x_n}(x) = [G(x; \hat{e}(x_1), z)]^n$$

(11)

It is clear that as the number of observation grows without bound,

$$\lim_{n \to \infty} F_{x_n}(x) = \begin{cases} 0 & \text{if } x \leq z \\ 1 & \text{if } x > z \end{cases}$$

(12)

almost surely. Recall that $z$ is the only unknown parameter of the true outcome distribution $G(x; e, z)$. This means that in the infinite horizon limit, even under the most pessimistic and unresponsive effort sequence $(\hat{e}(x_1), \hat{e}(x_1), \ldots)$, the agent eventually “discovers” or becomes aware of the true support of outcomes $[0, z]$ and therefore learns completely, in the sense of Smith and Sørensen (1997). If the effort sequence is instead optimal and therefore weakly increasing, the agents simply discover the interval $[0, z]$ faster.

4. Bayesian Updating with a Fixed Frontier

In this section, I assume that the actual frontier $z$ is drawn from some prior distribution and that the agents update beliefs in standard Bayesian fashion. This version of the model has similarities with that of Jovanovic and Nyarko (1995), who consider a micro-founded model of learning-by-doing in which the agent, simply by making decisions, learns more about some unknown parameter of his decision problem. Furthermore, the mathematical structure of this version of my model is similar to those found in Zeira (1987) and Rob (1991), but there are important differences which are discussed below.

To analyze the dynamics of the model under Bayesian updating, one needs to keep track of how beliefs evolve over time and to determine how beliefs influence optimal effort. The only sharp conclusion that can be drawn from Assumptions A1 and A3 is

\footnote{See e.g. Mood et al. (1974, Corollary to Theorem 11).}

\footnote{In fact, the sample distribution of outcomes converges to the true underlying distribution $G(x; \hat{e}(x_1), z)$ almost surely. See e.g. Parthasarathy (2005, Theorem 7.1).}

\footnote{The effort sequence $(\hat{e}(x_1), \hat{e}(x_1), \ldots)$ is unresponsive in the sense that it does not respond to the record $x_1$ actually being broken.}

\footnote{From a mathematical perspective, the Bayesian model studied in this section can be thought of as one of model uncertainty in the sense of Clyde and George (2004), where each element $z \in Z$ is a different model. Also, the model is technically one of estimation under random truncation, as studied by Woodroofe (1985).}
that the decision maker makes optimal choices given beliefs. In particular, there is a
priori no reason to expect any kind of monotonicity of either beliefs (in outcomes) or
in effort choices (in beliefs or in time). To see why this is so, recall that the outcome
x can be thought of as a noisy signal of the feasibility frontier z. The information that
is conveyed by the outcome thus depends on the information structure. In turn, this
structure determines belief formation and thereby optimal effort choices.

To make further progress, I therefore need to impose more structure on the model.
In order for posterior beliefs to be well behaved in the observed outcome x, I will
impose Assumption A2, i.e. the monotone likelihood ratio property. As shown by
Milgrom (1981), in its strict version this assumption implies that as the observed
outcome x increases, the posterior distribution over the feasibility frontier Z increases
in the sense of first-order stochastic dominance. That is, for any prior distribution
\( F(z) \) and outcomes \( x \) and \( x' \) with \( x > x' \), for every \( z > 0 \),

\[
F(z|x) \leq F(z|x')
\]

with strict inequality for at least some \( z \). Informally, observing a high outcome is
“good news” in the sense that the decision maker becomes more optimistic about the
distribution of the feasibility frontier Z.

As shown in Klemens (2007), A2 also implies that for all \( x \geq 0, e \geq 0 \) and \( z \geq z' \),

\[
G(x; e, z) \leq G(x; e, z')
\]

This simply means that the outcome distribution is stochastically increasing in the
possibility frontier \( z \).

Let the random frontier be denoted by \( Z \). In this section, I maintain the following
assumption:

\[ A6 \quad Z \sim F_1(z) \text{ on } \mathbb{R}_{++} \text{ with density } f_1(z). \]

Before writing up the agent’s problem, I describe the evolution of beliefs. For some
prior distribution \( F_1(z) \), the distribution at time \( t = 2, 3, ... \), upon having observed
outcome \( x \), is given by

\[
F_t(z|x) = \int_0^z f_t(z|x)dz
\]

where by Bayes’ rule, the updated probability density over feasibility frontiers \( Z \) upon
observing outcome \( x \) is given by

\[
f_t(z|x) = \frac{g(x; e, z)f_{t-1}(z)}{\int_{x}^{\infty} g(x; e, z)f_{t-1}(z)dz}
\]
The agent’s problem is then to solve

\[
\max_{e \geq 0} \left\{ \int_{0}^{\infty} \int_{0}^{z} u(x)g(x; e, z)f_t(z)dxdz - c(e) \right\}
\]  

(17)

The first-order condition for optimal effort \(e^*_t\) is

\[
\int_{0}^{\infty} \int_{0}^{z} u(x)g_e(x; e^*, z)f_t(z)dxdz - c_e(e^*) = 0
\]

(18)

From this expression, it is clear that the beliefs over possible frontiers \(z\) directly influence the expected marginal return to effort, and so can be expected to differ between naive and Bayesian updating.

To see how beliefs influence equilibrium effort, consider an increase in the belief distribution in the sense of first-order stochastic dominance (henceforth an FOSD increase). Recall that according to A4, the expected marginal return to effort under a known frontier \(z\) is increasing in \(z\). Therefore, an FOSD increase in beliefs increases the expected marginal return to effort. In order to restore the first order condition, effort must therefore increase (this follows from the second-order condition). In short, when the Bayesian agent becomes more optimistic about what is possible, he optimally increases his effort. Similarly, an FOSD decrease in beliefs causes him to optimally decrease effort. Summing up, under Bayesian updating, equilibrium effort \(e^*_t\) may be non-monotone over time. Note that this finding is relevant regardless of whether beliefs are monotone in outcomes. If Assumption A2 holds with strict inequality, then I obtain the following positive result:

**Proposition 3.** If the outcome distribution satisfies the strict monotone likelihood ratio property, then beliefs and optimal effort are increasing in observed outcomes.

Thus under Bayesian updating, equilibrium effort may be monotone in outcomes and in beliefs, but not necessarily monotone in time. In the model of Jovanovic and Nyarko (1995), each observation is equally informative and independent of the agent’s choice. In contrast, in this model the informational content of observations depends on the outcome, as emphasized in Section 5 below. Again, it should be emphasized that this monotonicity need not hold from period to period, as low outcomes (which are entirely possible) simply cause more pessimistic beliefs, which in turn decreases optimal effort. Similarly, a high observed outcome increases subsequent optimal effort.

\[19\] Note that by defining the reduced-form utility function

\[w(e, z) \equiv \int_{0}^{z} u(x)g(x; e, z)dx\]

where \(z\) is some unknown parameter, the agent’s optimization problem is entirely standard.

\[20\] The associated second-order condition for optimality is

\[\int_{0}^{\infty} \int_{0}^{z} u(x)g_{ee}(x; e^*, z)f_t(z)dxdz - c_{ee}(e^*) < 0\]

\[21\] This argument is also used by Rothschild and Stiglitz (1971).

\[22\] Under naive updating, equilibrium effort is also increasing in beliefs, but beliefs are monotone in time.
Next, I compare the equilibrium under Bayesian updating of a fixed frontier with the equilibrium under naive updating. In order to do so, it is useful to rewrite the naive agent’s objective function as an expectation over possible frontiers $Z$. This can be done by writing the problem as

$$
\max_{e \geq 0} \left\{ \int_0^\infty \int_0^z u(x)g(x; e, z)\delta(z - x)dx\,dz - c(e) \right\}
$$

(19)

where $\delta(s)$ is the Dirac delta function.\textsuperscript{23} That this problem is equivalent to the naive agent’s optimization problem, follows from the so-called sifting property, whereby for some function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$
\int_{-\infty}^{\infty} \varphi(s)\delta(s - k)ds = \varphi(k)
$$

(20)

The usefulness of rewriting the objective like this, stems from the fact that the cumulative distribution function of the Dirac delta function is the Heaviside function

$$
H(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases}
$$

(21)

This means that the naive objective function can be thought of as an expected value over the random variable $Z$ under the distribution function

$$
H(z - x_t) = \begin{cases} 1 & \text{if } z \geq x_t \\ 0 & \text{if } z < x_t \end{cases}
$$

(22)

With these observations in place, it is straightforward to compare the naive and Bayesian optimization problems. First, note that for any $s \geq t$,

$$
F_s(z) = 0
$$

(23)

for all $z < x_t$. It then follows that for all $z \in (0, \infty]$,

$$
F_t(z) \leq H(z - x_t)
$$

(24)

But this means that the beliefs under Bayesian updating first-order stochastically dominate the beliefs under naive updating. This is illustrated in Figure 2.

Because of A4, this means that the expected marginal return to effort under beliefs $F_t(z)$ are higher than under beliefs $H(z - x_t)$.\textsuperscript{24} Using the same arguments as above, I have therefore now established that

$$
e^*_t \geq \hat{e}_t
$$

(25)

\textsuperscript{23}The defining properties of the Dirac delta function are that $\delta(s) = 0$ for all $s \neq 0$ while $\int_{-\infty}^{\infty} \delta(s)ds = 1$. See Courant and John (1989).

\textsuperscript{24}Note that under naive updating, I concluded that effort was increasing in the record, i.e. that $\hat{e}(x_t) < \hat{e}(x_s)$ for $x_t < x_s$. This can also be seen by employing the present argument and noting the fact that $H(z - x_s) \leq H(z - x_t)$ for all $z$, i.e. an increase in the record induces an FOSD increase in the perceived distribution of $Z$ under naive updating.
I thus have the following result:

**Proposition 4.** Under Bayesian updating, equilibrium effort $e^*_t$ is higher than equilibrium effort $\hat{e}_t$ under naive updating, for all $t = 1, 2, \ldots$.

It is worth emphasizing that this result holds for any possible sequence of observations, even though the Bayesian agent’s posterior may move both rightward and leftward. In the extreme scenario when a very long sequence of low outcomes materializes, the Bayesian agent’s posterior comes closer and closer to that of the naive agent, but can never shift to the left of it.

There are a number of interesting differences between the settings with naive and Bayesian updating, even when the frontier $z$ is fixed. First, under naive updating, effort is always (weakly) increasing over time. This is because only good news prompt a change in beliefs (i.e. when a record is broken). Under Bayesian updating, effort depends on the agent’s beliefs, which change randomly over time as a function of the observed outcomes. Depending on the sequence of observations, posteriors can increase or decrease, thereby creating the possibility that the resulting effort sequence be non-monotone in time.

For both types of updating, a certain kind of truncation takes place as new information arrives. In particular, both the naive and the Bayesian agents agree that the frontier cannot fall in the interval $(0, \bar{x}_t)$. As records are broken over time, both agents thus repeatedly left-truncate the distribution of possible frontiers. But this is where the similarities end. For the naive agent, observations below the current record, i.e. $x_s \leq \bar{x}_t$ do not prompt any revision of beliefs, whereas the Bayesian agent uses such information to form a posterior as described above (but with an unchanged domain). In this sense, the naive agent’s updating is both more infrequent and more lumpy.

Turning to the long term behavior of the Bayesian learning process, when the frontier is fixed, the limiting beliefs and associated effort choices are independent of
the kind of updating the agent uses, i.e. naive or Bayesian. As described above, because of A3, all possible outcomes will be observed in the limit by the naive agent, almost surely. Thus he will eventually discover the entire domain of outcomes and hence make the myopically optimal effort choice. Similarly, under Bayesian updating, the agent eventually learns the precise frontier and therefore also makes the full information myopically optimal decisions. This is illustrated in Figure 3.

As records are broken over time, the naive agent’s beliefs (whose distribution function is characterized by the step function) move rightward until the jump coincides with the true frontier, i.e. $\bar{x}_t = z$. In turn, while the Bayesian agent’s expectations about the true frontier may move up and down, in the limit they must settle on the true value $z$, almost surely. Thus in the limit, the Bayesian agent’s beliefs are eventually characterized by a step function that coincides with that of the naive agent.

5. Bayesian Updating with a Growing Frontier

In the previous section, I assume that the frontier $z > 0$ is drawn from some distribution but then remains constant through time. In this section, I consider the possibility that the frontier has random (but unobserved) increases from one period to the next. This version of the model is similar in spirit to that of Zeira (1994).

I next make the following assumption:

**A7** The frontier evolves according to the law of motion $z_t - z_{t-1} = \nu_t$ for $t = 1, 2, \ldots$, with $\nu_t \sim \Lambda$ on $[0, \sigma]$ with $\sigma \in (0, \infty)$ and density $\lambda(\nu)$.

I furthermore assume that the innovations $\nu_t$ are identically and independently distributed over time and independent of $Z_{t-1}$. The distribution of $Z_t$ is then the convolution of the distributions of $Z_{t-1}$ and $\nu_t$, i.e.

$$\tilde{F}_t(z) = \int_0^\sigma F_t(z - \nu)\lambda(\nu)d\nu$$

(26)
where tilde is introduced to distinguish the distribution function from that under a fixed frontier $z$. It is straightforward that for all $t = 1, 2, \ldots$ and $z$, 

$$\tilde{F}_t(z) \leq F_t(z)$$ \hspace{2cm} (27)

so that the distribution under an increasing frontier first-order stochastically dominates the distribution under a fixed frontier. This is illustrated in Figure 4.

Using the argument that was employed in comparing the naive and Bayesian frameworks above, I conclude that, controlling for the prior distribution $F_{t-1}(z)$, 

$$\tilde{e}_t \geq e_t^*$$ \hspace{2cm} (28)

where $\tilde{e}_t$ is the optimal effort under an increasing frontier. I therefore have the following result:

**Proposition 5.** Controlling for prior beliefs, equilibrium effort $\tilde{e}_t$ under an increasing frontier is higher than equilibrium effort $e_t^*$ under a fixed frontier.

Note that in this setting, there are several important differences with the literature on demand uncertainty in the Zeira-Rob tradition. First, in the present model, outcomes are drawn randomly from a chosen distribution, rather than chosen directly. More importantly, in the present model, the agents never fully learn the true frontier in finite time (even when the frontier is fixed). In Zeira (1987) and Rob (1991), when the firm overshoots, i.e. chooses a capacity above the potential demand, it perfectly learns the demand and there is no residual uncertainty. Thus, unless potential demand continues to change, like in Zeira (1994), no further learning takes place. In the present analysis, even under a fixed frontier, the agent is never fully informed of its location.
Rather, upon observing an outcome, however large that outcome may be, the only inference that the agent can make is that this outcome is feasible, i.e. that it is below the actual frontier. Of course, in the infinite horizon limit the agent’s information becomes very precise and hence learning stops in this sense.

Knowing with certainty that the current frontier has been reached, as in Zeira (2004), has a dampening effect on effort that has no counterpart in the present model, at least not in the short run. In that model, knowing that the frontier has been reached is an unambiguous signal to the agent that higher effort is unlikely to be rewarded. In the present setting, the failure to break the existing record may be attributed to chance, since the outcomes are drawn from some distribution with full support (whatever effort level is chosen).

6. Social Optimality and Experimentation

Under naive belief updating and under Bayesian updating with myopic behavior, agents simply maximize current expected utility and disregard the possibility that current effort choices (and the resulting influence that these have on current outcomes) may have informational spillovers on future periods. In this section, I discuss the extension to socially optimal decision making, i.e. to the setting in which the agent explicitly engages in intertemporal expected utility maximization.

When an agent exerts effort, there are two distinct effects. First, the effort determines the distribution of this period’s outcome (which is all the agent cares about if he behaves myopically). Second, the outcome conveys information about the likely frontier \( z \). In other words, the current outcome influences the precision of the agent’s information. It is precisely this effect on the precision of future information that future agents (or the same agent in future periods, under forward-looking behavior) can benefit from. The intertemporal maximization problem can therefore be viewed as a problem of optimal experimentation. Since a myopic agent does not internalize this informational spillover on future decisions, he will tend to exert a suboptimal level of effort relative to the first best level, i.e. experiment too little, unless he has already learned all relevant information.\(^{25}\)

Consider the decision problem of a fully forward-looking agent, who maximizes discounted expected utility over an infinite horizon. This agent’s objective coincides with that of a benevolent social planner. Under a fixed frontier \( z \), the only thing that changes over time is the probability distribution \( F_t(z) \), i.e. the agent’s beliefs over what it is possible to achieve. It is therefore convenient to treat the agent’s beliefs as the state variable in his intertemporal optimization problem.

The expected beliefs (or the predictive distribution) in the next period, as a function of current data end effort, are given by the distribution function

\[
F_{t+1}(z|e) = \int_0^\infty \int_0^z F_{t+1}(z|g(x; e, z)f_{t}(z)dx dz
\]  

\(^{25}\)Problems of optimal experimentation are hard to analyze. Easley and Kiefer (1988) study a quite general setting and show, under very general assumptions on the primitives of the model, that the value function exists and is well-behaved. They further study the limiting properties of the learning process. Analysis of the limiting properties of related models is also found in Kiefer and Nyarko (1989) and in Aghion et al. (1991). Last, Kiefer (1989) and Wieland (2000) offer numerical solutions to learning models and are able to characterize optimal policies in tractable settings.
where the Bayesian posterior $F_{t+1}(z)$ is given by (15) and where an upper bar is introduced to distinguish realized beliefs from expected beliefs. Denote by $V(F_t(z))$ the value function when beliefs are $F_t(z)$, i.e.

$$V(F_t(z)) = \max_{e \geq 0} \left\{ \int_0^\infty \int_0^z u(x)g(x; e, z) f_t(z) dx dz - c(e) + \beta \int_0^\infty V(F_{t+1}(z)) d\overline{F}_{t+1}(z|e) \right\}$$

(30)

where $\beta \in (0, 1)$ is the discount factor. Returning to the agent’s optimization problem, he thus solves

$$\max_{e \geq 0} \left\{ \int_0^\infty \int_0^z u(x)g(x; e, z) f_t(z) dx dz - c(e) + \beta \int_0^\infty V(F_{t+1}(z)) d\overline{F}_{t+1}(z|e) \right\}$$

(31)

While the agent’s maximization problem is easily stated, it is notoriously difficult to characterize the optimal policy in this type of experimentation problem analytically. It is for this reason that researchers have either resorted to studying the limiting behavior of the learning process or employed numerical techniques.\(^{26}\)

It is straightforward that the solution to problem (31) in general differs from the solution to problem (17) under myopic decision making, unless the agent already has extremely precise information about the frontier $z$. In other words, the presence of a value of information that accrues only in the future, typically means that myopic behavior leads to distorted effort levels. The interesting question is thus not whether there is a distortion, but rather to determine the direction of the distortion. On the face of it, it may be tempting to conjecture that under intertemporal maximization, equilibrium effort is higher than under myopic behavior, ceteris paribus. It turns out that such a result is not immediate.

There are two possible ways in which intertemporal maximization may lead to higher effort than under myopic decision making. The first is to exploit the concavity of the value function in the space of beliefs. It is well known that regardless of risk attitudes, an agent would prefer experiments that lead to more dispersed beliefs (see e.g. Gollier, 2001 for details). As shown by Porta et al. (2006), the value function $V(F_t(z))$ is convex in beliefs $F_t(z)$. This means that any effort choice $e$ that induces a mean preserving spread in possible beliefs, would increase the expected value function accruing from choosing optimally in the next period onwards. The snag is that increasing effort does not unambiguously lead to a mean preserving spread in beliefs. While it is true that some effort levels lead to more information than others, it is not true in the present model that the informativeness of effort is monotone in the effort choice.\(^{27}\)

The second possible approach is to use results for monotone decision problems developed by Athey and Levin (2001). They show that a decision maker with supermodular preferences (in action and an exogenous parameter) would prefer information structures that induce posteriors that are ranked in the sense of first-order stochastic dominance. While the myopic problem (17) can immediately be analyzed using such

\(^{26}\)For a review of the literature and a discussion of the complications that arise in simulating such models, see Wieland (2000).

\(^{27}\)Of course, it is true that an exogenous mean preserving spread in the outcome distribution $G(x; e, z)$ will, for given effort $e$, increase the agent’s expected future utility.
techniques, the problem in applying them under intertemporal utility maximization is that it is difficult to ensure supermodularity of the value function in problem (31).

The upshot of these observations is that while it is generally true that myopic decision making leads to suboptimal effort choices and thus generate too little information, the direction of the distortion in effort choices is very difficult to determine (and may indeed change sign across the space of beliefs). I am therefore left to conclude with a rather negative result.

Denote by \( e_t^* \) the effort that solves the intertemporal maximization problem (31). Since the agent’s intertemporal objective is simply that of the myopic agent, plus some additional function of effort, it follows that \( e_t^{**} \neq e_t^* \). In other words, I have that controlling for prior beliefs, equilibrium effort \( e_t^{**} \) under forward-looking behavior is different from the equilibrium effort \( e_t^* \) under myopic behavior, unless the agent has learned the value of \( z \).

In conclusion, current effort has informational effects on the future which are ignored under myopic decision making, but experimentation does not seem to lead to monotonicity of effort (either in time or in beliefs).

Although I have not characterized the socially optimal policy, the main importance of considering the setting with intertemporal optimization is to gauge the consequences for the intertemporal distribution of records. It should be clear that experimentation would not in general lead to clusters in records as those generated under naive updating or under Bayesian but myopic behavior.

The version of the model with intertemporal optimization has an intimate relation to the literature on sequential sampling, as surveyed in DeGroot (2004). In the standard sequential sampling model, a decision maker must at each point in time decide whether to acquire costly information and make an informed decision or to make an uninformed decision and end the process. In essence, the agent trades off more precise information (which may lead to better decisions) against the cost of acquiring the additional information. In the present model, the choice is not whether to acquire additional information (all outcomes are informative), but rather a choice of how informative the new information is expected to be.

Regardless of this difference, the same kind of tradeoff is present here. To see this, note that because the second term of the objective (31) depends on current effort only through its effect on expected future beliefs, one can nicely decompose the agent’s objective into a term which reflects only (present) expected utility and a term that is purely informational. In particular, the rate of change in the second term can be interpreted as the value of information from increased effort. This value is gross of the costs of acquiring this information, which are in turn reflected in the shortfall in current maximized utility, brought about maximizing intertemporally rather than simply maximizing present expected utility.

Note that as the posterior becomes increasingly precise, the marginal return to information approaches zero. In the classical sequential sampling model, this means that the agent ceases sampling in finite time and makes a decision without acquiring further information. In the present model, actions and sampling decisions are a combined act and hence the decrease in the value of additional information gathering activity, is reflected in terms of the intertemporal motive for increasing effort dissipating. In the limit, as the frontier becomes known almost surely, the agent has no need for fur-
ther information gathering, which is the only difference between the myopic and the forward-looking objective functions. As a result, in the limit, the agent’s effort is the same whether he behaves myopically or maximizes social welfare. These observations are of course reflections of the limiting results obtained in the optimal experimentation literature cited above.

One of the interesting findings of this paper is that if the frontier is fixed, then under both naive, Bayesian/myopic and Bayesian/forward-looking behavior, beliefs and equilibrium efforts end up being the same (and socially optimal). Having said that, it should be emphasized that this result only holds in the limit and that on the transition path, effort levels and beliefs may vary greatly. In particular, the speed of convergence to the limit depends on the behavior of the agent. Slowest convergence occurs under naive updating because under this setting, effort is generally lower and thus the probability that the agent’s awareness expands will be lower. Under Bayesian updating with myopic a agent, convergence occurs faster because his effort is higher, which increases the probability of high (and record breaking) outcomes. The fastest convergence is that of the forward-looking Bayesian agent, because he actively seeks additional information over and above the level emanating from simple (current) expected utility maximization.

7. Conclusion

This paper has been couched in terms of utility maximization, discovery and learning. I would like to conclude by offering an alternative perspective on the setting I have developed, as one of motivation and the effects of role models. Paraphrasing Chung (2000), a role model is someone who (i) achieves something unusual and (ii) induces others to rationally try to excel. While the notion of role models is routinely used in discussions on social mobility, affirmative action and more generally as a tool for motivation in corporate and educational settings, only a few economists have dealt explicitly with it. Exceptions are Manski (1993) and Chung (2000), who analyze learning models similar in spirit to the one presented here. On the empirical side, Evans (1992) seems to be one of the few to estimate the effects of role models. He finds that the presence of a black teacher in the classroom increases the achievement of black students.

In the present setting, a role model can be viewed as any individual that breaks the mold and achieves something unlikely or unthinkable. Upon observing such feats, others are motivated to achieve. Judith Thomson, a proponent of affirmative action, has noted that “[...]black and women students do need role models, they do need concrete evidence that those of their race or sex can become accepted, successful, professionals - plainly, you won’t try to become what you don’t believe you can become.”

Of course, motivation is not a concept lost on coaches and motivators, who are well aware of the dichotomy between what is actually possible and what is perceived as possible. Myers (2002) paraphrases Tim Noakes, the world expert on endurance sports, and states that “however exhausted a runner may feel, he is likely to be a fair way from exhausting all his physical potential. The trick is to close the gap [...]”.

The analysis suggests that in many instances where individuals exert effort in the face of genuine uncertainty about what is feasible, they may likely benefit from

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28See Thomson (1973) and the discussion of her work in Chung (2000).
exogenous motivation that spurs them to try achieve the seemingly impossible.

APPENDIX

A. A Functional Form Example

In this appendix, I verify that under the stated assumptions, the outcome distribution function based on the normal distribution satisfies Assumption A4 for $z > 1$. The required derivatives are given by

\[ g_e(x; e, z) = \frac{\partial}{\partial e} \left( \frac{\lambda e}{x} \left( \frac{x}{z} \right)^{\lambda e} \right) = \lambda \left( \frac{x}{z} \right)^{\lambda e} \frac{\lambda e \ln z}{x} + 1 \]  \hspace{1cm} (32)

\[ g_{ez}(x; e, z) = -\frac{1}{xz} \lambda^2 e \left( \frac{x}{z} \right)^{\lambda e} \left( \lambda e \ln \frac{x}{z} + 2 \right) \]  \hspace{1cm} (33)

To verify that Assumption A4 holds under the sufficient condition, I need to calculate the following integral:

\[ I = \int_0^z \frac{1}{xz} \lambda^2 e \left( \frac{x}{z} \right)^{\lambda e} \left( \lambda e \ln \frac{x}{z} + 2 \right) \, dx \]  \hspace{1cm} (34)

In order to calculate this improper integral, I define

\[ I_v \equiv \int_0^z \frac{1}{xz} \lambda^2 e \left( \frac{x}{z} \right)^{\lambda e} \left( \lambda e \ln \frac{x}{z} + 2 \right) \, dx \]  \hspace{1cm} (35)

\[ = \left[ -\frac{\lambda^3 e^2}{z^{\lambda e+1}} \int_0^z \frac{x^{\lambda e}}{\lambda e+1} \ln x \, dx + \left( \frac{\lambda^3 e^2}{z^{\lambda e+1}} \ln z - \frac{2\lambda^2 e}{z^{\lambda e+1}} \right) \int_0^z x^{\lambda e} \, dx \right] \]  \hspace{1cm} (36)

\[ = \frac{\lambda^3 e^2}{\lambda e+1} \left[ \left( \frac{\lambda e+1}{\lambda e} \ln v - \ln z \right) - \left( \frac{1}{\lambda e+1} \ln v + 1 - \frac{1}{\lambda e+1} \right) \right] \]

\[ + \frac{\lambda^3 e^2 \ln z - 2\lambda^2 e}{\lambda e+1} \left( 1 - \left( \frac{\lambda e}{z} \right)^{\lambda e+1} \right) \]  \hspace{1cm} (37)

where I have used integration by parts to calculate the first integral in the bracket.

The last step is to evaluate the integral $I_v$ in the limit as $v \to 0$. Using L'Hôpital’s rule, it follows that

\[ \lim_{v \to 0} I_v = \frac{\lambda^2 e}{\lambda e+1} \left( \frac{\lambda e}{\lambda e+1} \frac{\lambda e}{z^{\lambda e+1}} - 2 \right) \]  \hspace{1cm} (38)

Assumption A4 requires that

\[ I + zg_e(z; e, z) > 0 \]  \hspace{1cm} (39)

Substituting for the relevant terms, this inequality becomes

\[ I + zg_e(z; e, z) = \frac{\lambda^2 e}{\lambda e+1} \left( \frac{\lambda e}{\lambda e+1} \frac{\lambda e}{z^{\lambda e+1}} - 2 \right) + \lambda \]  \hspace{1cm} (40)

\[ = \frac{\lambda}{(\lambda e + 1)^2} \left( \lambda^2 e^2 (z^{\lambda e+1} - 1) + 1 \right) > 0 \]  \hspace{1cm} (41)

where the inequality holds if $z > 1$. Note that this is only a sufficient condition.
weaker sufficient condition is that
\[ z > \left(1 - \frac{1}{\lambda^2 e^2}\right)^{\frac{1}{\lambda e^2 + 1}} \]  
(42)
where the right-hand is strictly smaller than one.

References


