THE OPTIMAL CONTROL OF INFECTIOUS DISEASES VIA PREVENTION AND TREATMENT: Supplementary Material and Omitted Proofs.

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ABSTRACT. In Section 1, we present the parameter restrictions for the different steady states to be feasible. In Section 2, we prove the existence of an optimal solution. In Section 3, we analyze the dynamics around the fully interior steady states, characterize spiral sources and prove that spiraling is non-optimal. In Section 4 we outline the model with imperfect protection. In Section 5, we discuss the issues of substitutes and complements, most rapid approach paths and speeds of convergence. In Section 6, we present results on comparative dynamics and welfare. In Section 7, we analyze the local stability of equilibrium steady states. In Section 8, we offer additional characterization of the maverick's problem. In Section 9, we verify the non-sufficiency of the Hamiltonian conditions for the planner's problem. In Section 10, we set out a model of optimal quarantines.

1. PARAMETER RESTRICTIONS FOR STEADY STATES IN CENTRALIZED SETTING

Throughout this paper, we have maintained the assumption that $\omega - c_P > 0$ and $\beta - \gamma - \alpha > 0$. In this appendix, we list additional assumptions that ensure that the different fixed points are feasible.

1.1. Fixed Point A.

For this steady state to be feasible, we need the following additional restrictions:

- For $I(t) \in (0, 1)$ need $c_P < \frac{\beta\omega}{\rho+\beta}$.
- For $\lambda(t) < 0$ need $c_P < \omega$.
- For $\pi(t) \in (0, 1)$ need $c_P < \omega$.
- For $\tau(t) = 0$ need $c_P > \omega - c_T \left(\frac{\rho}{\alpha}\right)$.

1.2. Fixed Point B.

For this steady state to be feasible, we need the following additional restrictions:

- For $I(t) \in (0, 1)$ need $c_P < \frac{\beta(\omega+c_T)}{\rho+\beta}$.
- For $\lambda(t) < 0$ need $c_P < \omega + c_T$.
- For $\pi(t) \in (0, 1)$ need $c_P < \left(\frac{\beta-\alpha-\gamma}{\beta-\alpha-\gamma+\rho}\right)(\omega + c_T)$.
- For $\tau(t) = 1$ need $c_P < \omega + c_T \left(\frac{\alpha-\rho}{\alpha}\right)$.

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1.3. Fixed Point C. For this steady state to be feasible, we need the following additional restrictions:

- For $I(t) \in (0,1)$ need $c_P < \frac{\beta c_T}{\alpha}$.
- For $\lambda(t) < 0$, no extra restriction.
- For $\pi(t) \in (0,1)$ need $c_P < \min\left\{\frac{\omega}{2} + c_T\left(\frac{\beta-\gamma-\rho}{2\alpha}\right), \frac{\beta c_T}{\alpha}\right\}$.
- For $\tau(t) \in (0,1)$ need $c_P \in \left(\omega - c_T\left(\frac{\rho}{\alpha}\right), \omega + c_T\left(\frac{\alpha-\rho}{\alpha}\right)\right)$.

1.4. Fixed Point A_0. For this steady state to be feasible, we need the following additional restrictions:

- For $I(t) \in (0,1)$ need $\beta > \gamma$.
- For $\lambda(t) < 0$ need $\beta > \gamma - \rho$.
- For $\pi(t) = 0$ need $c_P > \frac{\omega(\beta-\gamma)}{\beta-\gamma+\rho}$.
- For $\tau(t) = 0$ need $c_T > \frac{\alpha\omega}{\beta-\gamma+\rho}$.

1.5. Fixed Point B_0. For this steady state to be feasible, we need the following additional restrictions:

- For $I(t) \in (0,1)$, no extra restriction.
- For $\lambda(t) < 0$, no extra restriction.
- For $\pi(t) = 0$ need $c_P > \frac{(\omega+c_T)(\beta-\gamma-\alpha)}{\beta-\gamma+\rho-\alpha}$.
- For $\tau(t) = 1$ need $c_T < \frac{\alpha\omega}{\beta-\gamma+\rho-2\alpha}$.

1.6. Fixed Point C_0. For this steady state to be feasible, we need the following additional restrictions:

- For $I(t) \in (0,1)$, need $c_T \in \left(\frac{\alpha\omega}{\beta+\gamma+\rho}, \frac{\alpha\omega}{\gamma+\rho-\beta}\right)$.
- For $\lambda(t) < 0$, no extra restriction.
- For $\pi(t) = 0$ need $c_P > \frac{\alpha\omega + c_T(\beta-\gamma-\rho)}{2\alpha}$.
- For $\tau(t) \in (0,1)$ need $c_T \in \left(\frac{\alpha\omega}{\beta-\gamma+\rho}, \frac{\alpha\omega}{\beta-\gamma+\rho-2\alpha}\right)$.

2. Existence of an Optimal Solution

In this appendix, we prove that the planner's problem admits an optimal solution. The formal result is as follows:

Theorem 1. *An optimal solution $(I^*(t), \tau^*(t), \pi^*(t))$ exists if at least one of the fixed points A, B, A_0, B_0 (to be specified below) is feasible.*

Proof: The existence proof proceeds in two steps. In Step 1, we consider finite horizon versions of the model and show that in these, an optimal solution exists. In Step 2, we show by contradiction that because optimal solutions exists for all finite horizons, an optimal solution must also exist for the infinite horizon version.
Step 1: Consider a finite horizon version of the model in which $t \in [0, T]$, with $T < \infty$. Define the set

$$N(I, U, t) \equiv \left\{e^{-\rho t}\left(-\omega I - c_P(1-I)\pi - c_T I\tau\right) + \xi, I\left(\beta(1-I)(1-\pi) - \gamma - \alpha\tau\right) : (\tau, \pi) \in U\right\} \tag{1}$$

where $\xi \leq 0$ is some constant and $U = [0,1] \times [0,1]$ is the space of feasible control pairs.
Consider two points $y_1, y_2 \in N(I, U, t)$ given by

$$y_1 \equiv \left\{e^{-\rho t}\left(-\omega I - c_P(1-I)\pi_1 - c_T I\tau_1\right) + \xi_1, I\left(\beta(1-I)(1-\pi_1) - \gamma - \alpha\tau_1\right)\right\} \tag{2}$$

$$y_2 \equiv \left\{e^{-\rho t}\left(-\omega I - c_P(1-I)\pi_2 - c_T I\tau_2\right) + \xi_2, I\left(\beta(1-I)(1-\pi_2) - \gamma - \alpha\tau_2\right)\right\} \tag{3}$$

Let $\varphi \in [0,1]$ and let $y_3 \equiv \varphi y_1 + (1-\varphi)y_2$. We will prove that $y_3 \in N(I, U, t)$ and thus that the set $N(I, U, t)$ is convex. Let $\varphi y_1 + (1-\varphi)y_2 = (z_1, z_2)$. Taking the first element, we have that

$$\begin{aligned} z_1 &= \varphi\left[e^{-\rho t}\left(-\omega I - c_P(1-I)\pi_1 - c_T I\tau_1\right) + \xi_1\right] \\ &\quad +(1-\varphi)\left[e^{-\rho t}\left(-\omega I - c_P(1-I)\pi_2 - c_T I\tau_2\right) + \xi_2\right] && (4) \\ &= e^{-\rho t}\left(-\omega I - c_P(1-I)\pi_3 - c_T I\tau_3\right) + \xi_3 && (5) \end{aligned}$$

where $\tau_3 \equiv \varphi\tau_1 + (1-\varphi)\tau_2$, $\pi_3 \equiv \varphi\pi\tau_1 + (1-\varphi)\pi_2$ and $\xi_3 \equiv \varphi\xi_1 + (1-\varphi)\xi_2 \leq 0$.
Similarly, taking the second element we have that

$$\begin{aligned} z_2 &= \varphi\left[I\left(\beta(1-I)(1-\pi_1) - \gamma - \alpha\tau_1\right)\right] \\ &\quad +(1-\varphi)\left[I\left(\beta(1-I)(1-\pi_2) - \gamma - \alpha\tau_2\right)\right] && (6) \\ &= I\left[\beta(1-I)(1-\pi_3) - \gamma - \alpha\tau_3\right] && (7) \end{aligned}$$

We can now conclude that: (i) there exist an admissible triple $(I(t), \tau(t), \pi(t))$; (ii) the set $N(I, U, t)$ is convex for each $(I(t), t)$; (iii) the set U is closed and bounded; (iv) there exists a bound $b = 1$ such that $\|I(t)\| < b$ for all $t \geq 0$ and admissible triples $(I(t), \tau(t), \pi(t))$. By the Filippov-Cesari Theorem, we can then conclude that an optimal solution $(I^*(t), \tau^*(t), \pi^*(t))$ exists and the optimal policy $(\tau^*(t), \pi^*(t))$ is measurable. See Seierstad and Sydsaeter (1987) for details.
Step 2: We will consider the case in which the relevant steady states are (A, B). The case (A_0, B_0) follows similar steps. In the finite horizon version of our problem, we impose no condition on the terminal value $I(T)$. This implies that the relevant transversality

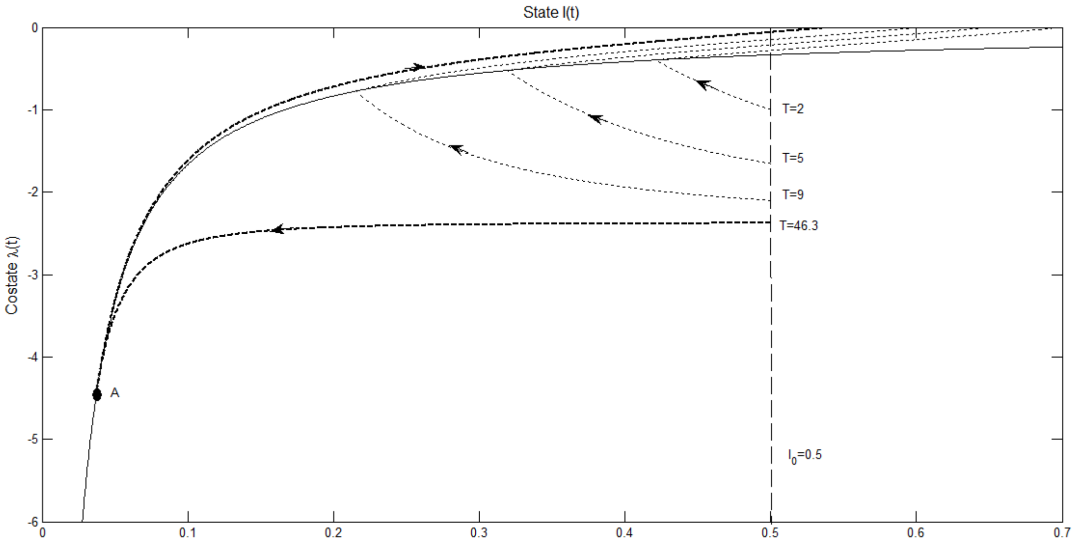

Figure 1: Paths in finite horizon model around point A.

condition is $\lambda(T) = 0$. As we have shown, this problem has an optimal solution. Moreover, if T is large enough, there are at at most two candidates for an optimum. Each path satisfies the necessary conditions for optimality, including the aforementioned transversality condition. Of these two candidate optimal paths, one goes to solution A as in the infinite horizon case but then at time $t = T - T_A$, peels off along the unstable branch to increase monotonically, reaching $\lambda(t) = 0$ at time $t = T$. The other path goes to solution B as in the infinite horizon case, but then at time $t = T - T_B$, peels off along the unstable branch to increase monotonically, reaching $\lambda(t) = 0$ at time $t = T$. Note that the times T_A and T_B are fixed.

In Figure 1, we illustrate the idea by plotting optimal paths around the point A. The parameters are the same as those in Example 1. The optimal paths from initial condition $I_0 = 0.5$ and different horizons T are represented by dashed curves. Note that the light paths reach the $\lambda(T) = 0$ line faster than the heavy dashed path that goes through point A, since the latter stays at point A until time $t = T - T_A$ regardless of how long it took that path to reach point A. In contrast, the light dashed paths do not rest at any point until they reach their destination. For all horizons $T \geq 46.3$, the optimal path reaches point A before making the transition to the $\lambda(T) = 0$ line, while for horizons $T < 46.3$, the point A is not reached along an optimal path. While we have shown only a case where $I_0 > I_A$, similar analysis applies for the case $I_0 < I_A$. Similar analysis also applies for optimal finite horizon paths in the vicinity of the other steady states.

From the point A, there is a unique path satisfying the Hamiltonian conditions and starting from $(I(0), \lambda(0)) = (I_A, \lambda_A)$ and is such that $\lambda(T_A) = 0$ for some $T_A > 0$. The time T_A is unique. Denote the value of the integral along this path as follows:

$$W_A \equiv \int_0^{T_A} e^{-\rho t} \left[I(t) \left[\omega_{\mathcal{I}} - c_T \tau(t) \right] + (1 - I(t)) \left[\omega_{\mathcal{S}} - c_P \pi(t) \right] \right] dt \tag{8}$$

From the point B, there is similarly a unique path satisfying the Hamiltonian conditions and starting from $(I(0), \lambda(0)) = (I_B, \lambda_B)$ and is such that $\lambda(T_B) = 0$ for some $T_B > 0$. The time T_B is unique. Denote the value of the integral along this path as follows:

$$W_B \equiv \int_0^{T_B} e^{-\rho t} \left[I(t) \left[\omega_{\mathcal{I}} - c_T \tau(t) \right] + (1 - I(t)) \left[\omega_{\mathcal{S}} - c_P \pi(t) \right] \right] dt \tag{9}$$

From now on, we shall consider only paths that begin at $I(0) = I_0$. Let

$$V_A^{\overline{T}} \equiv \int_0^{\overline{T}} e^{-\rho t} \left[I(t) \left[\omega_{\mathcal{I}} - c_T \tau(t) \right] + (1 - I(t)) \left[\omega_{\mathcal{S}} - c_P \pi(t) \right] \right] dt \tag{10}$$

where the integral is evaluated along the Hamiltonian path and terminates at point A at time $\overline{T}$. Also, let

$$V_B^{\overline{T}} \equiv \int_0^{\overline{T}} e^{-\rho t} \left[I(t) \left[\omega_{\mathcal{I}} - c_T \tau(t) \right] + (1 - I(t)) \left[\omega_{\mathcal{S}} - c_P \pi(t) \right] \right] dt \tag{11}$$

where the integral is evaluated along the Hamiltonian path and terminates at point B at time $\overline{T}$.

Finally, let

$$X_A^T \equiv \int_0^T e^{-\rho t} \left[I(t) \left[\omega_{\mathcal{I}} - c_T \tau(t) \right] + (1 - I(t)) \left[\omega_{\mathcal{S}} - c_P \pi(t) \right] \right] dt \tag{12}$$

where the integral is evaluated along the Hamiltonian path that goes to point A and sits there until time $t = T - T_A$ and then peels off to reach $\lambda(t) = 0$ at time $t = T$. Also, let

$$X_B^T \equiv \int_0^T e^{-\rho t} \left[I(t) \left[\omega_{\mathcal{I}} - c_T \tau(t) \right] + (1 - I(t)) \left[\omega_{\mathcal{S}} - c_P \pi(t) \right] \right] dt \tag{13}$$

where the integral is evaluated along the Hamiltonian path that goes to point B and sits there until time $t = T - T_B$ and then peels off to reach $\lambda(t) = 0$ at time $t = T$.

It is clear that

$$X_A^T = V_A^{T-T_A} + e^{-\rho(T-T_A)} W_A \tag{14}$$

$$X_B^T = V_B^{T-T_B} + e^{-\rho(T-T_B)} W_B \tag{15}$$

Suppose without loss of generality that in the infinite horizon case, it is better to go to point B and stay there than to go to point A and stay there. Then

$$\lim_{T\to\infty} V_B^{T-T_B} = V_B^{\infty} > V_A^{\infty} = \lim_{T\to\infty} V_A^{T-T_A} \tag{16}$$

From the above equations, it then follows that

$$\lim_{T\to\infty} X_B^T > \lim_{T\to\infty} X_A^T \tag{17}$$

Thus, in the finite horizon case, it is optimal for large T to go to point B and then peel off at time $t = T - T_B$.

Suppose there is no optimal path in the infinite horizon case. Then there is some path starting from $I(0) = I_0$ for which the value of the integral is greater than V_B^{∞}. Let

$$Z^T \equiv \int_0^T e^{-\rho t} \left[I(t) \left[\omega_{\mathcal{I}} - c_T \tau(t) \right] + (1 - I(t)) \left[\omega_{\mathcal{S}} - c_P \pi(t) \right] \right] dt \tag{18}$$

where the integral is evaluated along this alternative path.

By assumption,

$$\lim_{T\to\infty} Z^T = Z^{\infty} > V_B^{\infty} = \lim_{\overline{T}\to\infty} V_B^{\overline{T}} \tag{19}$$

This implies that there exist $\overline{T}^*, T^*, \varepsilon > 0$ such that for all $\overline{T} > \overline{T}^*$and $T > T^*$, the inequality $Z^T > V_B^{\overline{T}} + \varepsilon$ holds. Hence, for $T > \max\left\{\overline{T}^* + T_B, T^*\right\}$, it follows that

$$Z^T > V_B^{T-T_B} + \varepsilon \tag{20}$$

Now, for sufficiently large T, $\varepsilon > e^{-\rho(T-T_B)} W_B$ and hence

$$Z^T > V_B^{T-T_B} + e^{-\rho(T-T_B)} W_B = X_B^T \tag{21}$$

But this is not possible, since X_B^T is optimal. This contradiction establishes that there must be an optimal solution to the infinite horizon problem. This concludes the proof ■

3. Analysis of Interior Solutions

In this appendix, we formally analyze the optimality properties and dynamic behavior around interior steady states. We do so through a sequence of different results.

3.1. Non-Optimality of Points C and C_0.

Next, we will show the following result:

Proposition 2. *No optimal path converges to either C or C_0.*

To formally establish the non-optimality of the interior points C and C_0, we first prove a useful relationship between the value function and the Hamiltonian. This part of the proof is related to a result by Mäler et al. (2003), but theirs applies only to fully interior controls and we must therefore make suitable changes and exploit that controls are constant almost everywhere along optimal paths.[1]

Lemma 3. $\rho V(I_0) = H^C(I_0, \tau(0), \pi(0), \lambda(0))$.

Proof: Consider a path which starts from the point $I(0) = I_0$, for which the control variables $\tau(t)$ and $\pi(t)$ are piecewise continuous and which satisfies the first order Hamiltonian conditions. For any path that satisfies these conditions together with the transversality condition and the laws of motion for state and costate variables, the following are true:
(1) Suppose that

$$-\left[c_T + \alpha\lambda(t)\right] I(t) < 0 \tag{22}$$

Then $\tau(t) = 0$ is optimal. Since $\lambda(t)$ and $I(t)$ are continuous along the path in question, it follows that

$$-\left[c_T + \alpha\lambda(t+\varepsilon)\right] I(t+\varepsilon) < 0 \tag{23}$$

for sufficiently small $\varepsilon > 0$ and hence $\tau(t+\varepsilon) = 0$. Thus, $d\tau(t)/dt = 0$ at time t. Likewise, $d\tau(t)/dt = 0$ if

$$-\left[c_T + \alpha\lambda(t)\right] I(t) > 0 \tag{24}$$

which makes $\tau(t) = 1$ optimal. Finally, if

$$\left[c_T + \alpha\lambda(t)\right] I(t) = 0 \tag{25}$$

then the Hamiltonian is independent of the treatment rate and therefore $\partial H^C/\partial\tau(t) = 0$. Thus, it is always the case that

$$\frac{\partial H^C}{\partial\tau(t)}\frac{d\tau(t)}{dt} = 0 \tag{26}$$

(2) Suppose

$$-\left[c_P + \beta\lambda(t)I(t)\right](1 - I(t)) < 0 \tag{27}$$

Then $\pi(t) = 0$ is optimal. Since $\lambda(t)$ and $I(t)$ are continuous along the path in question, it follows that

$$-\left[c_P + \beta\lambda(t+\varepsilon)I(t+\varepsilon)\right](1 - I(t+\varepsilon)) < 0 \tag{28}$$

[1] We have an alternative proof of the non-optimality of the interior points, but the present derivation is more elegant.

for sufficiently small ε and hence $\pi(t+\varepsilon)=0$. Thus, $d\tau(t)/dt=0$ at time t. Likewise, $d\tau(t)/dt=0$ if

$$-\left[c_P+\beta\lambda(t+\varepsilon)I(t+\varepsilon)\right](1-I(t+\varepsilon))>0 \tag{29}$$

which makes $\pi(t)=1$ optimal. Finally, if

$$\left[c_P+\beta\lambda(t)I(t)\right](1-I(t))=0 \tag{30}$$

then the Hamiltonian is independent of the prevention rate and therefore $\partial H^C/\partial\pi(t)=0$. Thus, it is always the case that

$$\frac{\partial H^C}{\partial\pi(t)}\frac{d\pi(t)}{dt}=0 \tag{31}$$

The current-value Hamiltonian H^C is a function of $I(t),\lambda(t),\tau(t)$ and $\pi(t)$. Hence totally differentiating the current-value Hamiltonian yields

$$\begin{aligned}
\frac{dH^C}{dt} &= \frac{\partial H^C}{\partial I(t)}\frac{dI(t)}{dt}+\frac{\partial H^C}{\partial\lambda(t)}\frac{d\lambda(t)}{dt}+\frac{\partial H^C}{\partial\tau(t)}\frac{d\tau(t)}{dt}+\frac{\partial H^C}{\partial\pi(t)}\frac{d\pi(t)}{dt} &(32)\\
&= \frac{\partial H^C}{\partial I(t)}\frac{dI(t)}{dt}+\frac{dI(t)}{dt}\left(\rho\lambda(t)-\frac{\partial H^C}{\partial I(t)}\right)+\frac{\partial H^C}{\partial\tau(t)}\frac{d\tau(t)}{dt}+\frac{\partial H^C}{\partial\pi(t)}\frac{d\pi(t)}{dt} &(33)\\
&= \rho\lambda(t)\frac{dI(t)}{dt} &(34)
\end{aligned}$$

where we have used that

$$\begin{aligned}
\dot{I}(t) &= \frac{\partial H^C}{\partial\lambda(t)} &(35)\\
\dot{\lambda}(t) &= \rho\lambda(t)-\frac{\partial H^C}{\partial I(t)} &(36)
\end{aligned}$$

Next, we have that

$$\begin{aligned}
\frac{d(e^{-\rho t}H^C)}{dt} &= \rho e^{-\rho t}\left[-H^C+\lambda(t)\frac{dI(t)}{dt}\right] &(37)\\
&= \rho e^{-\rho t}\left[\omega I(t)+c_P\pi(t)(1-I(t))+c_T\tau(t)I(t)\right] &(38)
\end{aligned}$$

Since the transversality condition $\lim_{t\to\infty}e^{-\rho t}H^C(t)=0$ must hold, it follows that

$$\int_0^\infty\left[\frac{d(e^{-\rho t}H^C)}{dt}\right]dt=\lim_{t\to\infty}e^{-\rho t}H^C(t)-H^C(0)=-H^C(0) \tag{39}$$

Thus

$$\begin{aligned}
H^C(x_0,u(0),\lambda(0)) &= -\int_0^\infty\left[\frac{d(e^{-\rho t}H^C)}{dt}\right]dt &(40)\\
&= -\rho\int_0^\infty e^{-\rho t}\left(\omega I(t)+c_P\pi(t)(1-I(t))+c_T\tau(t)I(t)\right)dt &(41)\\
&= \rho V(I_0) &(42)
\end{aligned}$$

Hence

$$\rho V(I_0) = H^C(I_0, \tau(0), \pi(0), \lambda(0)) \tag{43}$$

This completes the proof ■

We now turn to the proof of the non-optimality of the interior solutions. Suppose there is a path that starts at $I(0) = I_C$ and has

$$\lambda(0) = \lambda_C^* > \lambda_C = \frac{-c_T}{\alpha} \tag{44}$$

and hence $\beta\lambda_C^* I_C > \beta\lambda_C I_C = -c_P$. The first inequality implies that $\tau(t) = 0$ is optimal and the second implies that $\pi(t) = 0$ is optimal. Equation (43) implies that value of the integral for the stationary path that remains at I_C is given by

$$\begin{aligned} \rho V_C &= -\omega - c_P\pi_C(1 - I_C) - c_T\tau_C I_C + \lambda_C I_C\left[(1 - \pi_C)\beta(1 - I_C) - \gamma - \alpha\tau_C\right] && (45) \\ &= -\omega + \lambda_C I_C\left[\beta(1 - I_C) - \gamma\right] && (46) \end{aligned}$$

The value of the integral along the alternative path is found by setting $I(0) = I_C$, $\lambda(0) = \lambda_C^*$, $\tau(0) = 0$ and $\pi(0) = 0$. Using (43), this yields the following expression for the integral along this path:

$$\rho V^* = -\omega + \lambda_C^* I_C\left[\beta(1 - I_C) - \gamma\right] \tag{47}$$

By subtraction,

$$\rho(V^* - V_C) = (\lambda_C^* - \lambda_C)I_C\left[\beta(1 - I_C) - \gamma\right] \tag{48}$$

Note that $\dot{I}(t) = 0$ if $I(t) = I_C, \tau(t) = \tau_C, \pi(t) = \pi_C$. Hence

$$\dot{I}(t) = I_C\left[(1 - \pi_C)\beta(1 - I_C) - \gamma - \tau_C\alpha\right] = 0 \tag{49}$$

Since $I_C, \tau_C, \pi_C > 0$, if follows that

$$I_C\left[\beta(1 - I_C) - \gamma\right] > I_C\left[(1 - \pi_C)\beta(1 - I_C) - \gamma - \tau_C\alpha\right] = 0 \tag{50}$$

Since $\lambda_C^* > \lambda_C$, it follows that $V^* > V_C$. Thus it is better to choose the alternative path than to remain at C. These arguments also apply to the point C_0. This concludes the proof ■

3.2. Optimal Paths, Spiral Sources and Limit Cycles. Although the interior points C and C_0 cannot be end points of optimal paths, it is necessary to consider the behavior of paths starting at these points. Our simulations show that such paths may be spirals, but formally showing that this is the case is complicated by the fact that standard results for the local behavior around such points do not apply to our problem. This is due to the discontinuities in the optimal policies in steady state. In characterizing the candidate solutions for optimal paths, there is a further potential complication, namely the possibility that the paths close to the interior steady states constitute limit cycles (i.e. closed orbits around the interior point). We will now show two results. First, we show that the interior solutions are indeed spiral sources, i.e. exploding spirals. We prove this result by appealing to a theorem due to Wagener (2003), which excludes limit cycles. We then extend his reasoning to exclude that the interior points are spiral sinks. By implication, the points must be spiral sources. Second, having established the spiraling

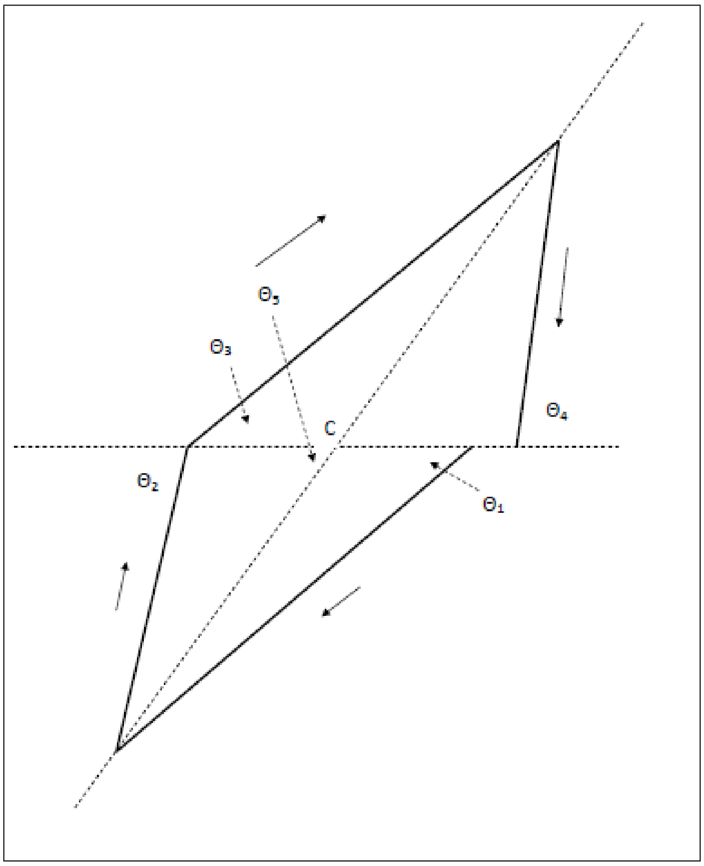

Figure 2: Rotation around interior solution C with linearized system.

nature of paths originating at the interior solutions, we characterize the candidate optimal paths.

Proposition 4. *The points C and C_0 are clock-wise spiral sources.*

Proof: The proof is in two parts. First, we prove that the movement around the interior solutions is characterized by clock-wise rotation. Second, we show that the movement is necessarily an exploding spiral.

Rotation Around Interior Solutions. We now prove that the movement around the interior points is a clock-wise rotation. Suppose that the interior stationary solution is C. The diagram in Figure 2 shows a linearized segment of a path in the vicinity of C and the angles θ_i, $i = 1, ..., 5$. We shall now show that

$$\begin{aligned} 90^0 &> \theta_1, \theta_3, \theta_4, \theta_5 > 0 && (51) \\ 180^0 &> \theta_2 > 0 && (52) \\ 90^0 &> \theta_1 - \theta_5, \theta_5 - \theta_3, \theta_4 - \theta_5 > 0 && (53) \\ 180^0 &> \theta_2 - \theta_5 > 0 && (54) \end{aligned}$$

Let $t_i = \tan\theta_i$. Then it follows that

$$t_i = \frac{\dot{\lambda}(t)}{\dot{I}(t)} \tag{55}$$

For $i = 1, ..., 4$, the rates of change $\dot{I}(t)$ and $\dot{\lambda}(t)$ are calculated by choosing the appropriate values of $\tau(t)$ and $\pi(t)$ and inserting the equilibrium values I_C and λ_C into the laws of motion for the state and costate variables, i.e.

$$\dot{I}(t) = I(t)\left[(1-\pi(t))\beta(1-I(t)) - \gamma - \alpha\tau(t)\right] \tag{56}$$

$$\dot{\lambda}(t) = \lambda(t)\left[\rho+\gamma+\alpha\tau(t)+(1-\pi(t))\beta(2I(t)-1)\right] + \left[\omega - \pi(t)c_P + \tau(t)c_T\right] \tag{57}$$

We now proceed to consider each angle in turn:

Angle θ_1: $\tau(t) = 1, \pi(t) = 1$. This yields the laws of motions

$$\begin{aligned}
\dot{I}(t) &= I_C\left[-\gamma-\alpha\right] && (58)\\
&= -\frac{\alpha c_P}{\beta c_T}(\gamma+\alpha) < 0 && (59)\\
\dot{\lambda}(t) &= \lambda_C\left[\rho+\gamma+\alpha\right] + \left[\omega - c_P + c_T\right] && (60)\\
&= -\frac{c_T}{\alpha}\left(\rho+\gamma\right) + \left(\omega - c_P\right) < 0 \text{ if } C \text{ is allowable} && (61)
\end{aligned}$$

and hence

$$\begin{aligned}
t_1 &= \frac{\dot{\lambda}(t)}{\dot{I}(t)} && (62)\\
&= \frac{\frac{c_T}{\alpha}\left(\rho+\gamma\right) - \left(\omega - c_P\right)}{\frac{\alpha c_P}{\beta c_T}(\gamma+\alpha)} > 0 && (63)
\end{aligned}$$

Thus, $90^0 > \theta_1 > 0$.

Angle θ_2: $\tau(t) = 1, \pi(t) = 0$. This yields the laws of motion

$$\begin{aligned}
\dot{I}(t) &= I_C\left[\beta(1-I_C) - \gamma - \alpha\right] && (64)\\
&= \frac{\alpha c_P}{\beta c_T}\left[\beta\left(1 - \frac{\alpha c_P}{\beta c_T}\right) - \gamma - \alpha\right] && (65)
\end{aligned}$$

$$\begin{aligned}
\dot{\lambda}(t) &= \lambda_C\left[\rho+\gamma+\alpha+\beta(2I_C-1)\right] + \left[\omega + c_T\right] && (66)\\
&= -\frac{c_T}{\alpha}\left[\rho+\gamma-\beta\right] + \left[\omega - 2c_P\right] > 0 \text{ if } C \text{ is allowable} && (67)
\end{aligned}$$

and hence

$$t_2 = \frac{\dot{\lambda}(t)}{\dot{I}(t)} \tag{68}$$

$$= \frac{-\frac{c_T}{\alpha}\left[\rho+\gamma-\beta\right]+\left[\omega-2c_P\right]}{\frac{\alpha c_P}{\beta c_T}\left[\beta\left(1-\frac{\alpha c_P}{\beta c_T}\right)-\gamma-\alpha\right]} \tag{69}$$

Since $\dot{\lambda}(t) > 0$, it follows that $180^0 > \theta_2 > 0$.
Angle θ_3: $\tau(t) = 0, \pi(t) = 0$. This yields the law of motion for prevalence as

$$\dot{I}(t) = I_C\left[\beta(1-I_C)-\gamma\right] \tag{70}$$

$$= \frac{\alpha c_P}{\beta c_T}\left[\beta\left(1-\frac{\alpha c_P}{\beta c_T}\right)-\gamma\right] > 0 \text{ since } I_C < I_{A_0} = \frac{\beta-\gamma}{\beta} \tag{71}$$

Note that $I(t)$ converges to $\frac{\beta-\gamma}{\beta}$ if there is no treatment or protection. Since there is some treatment and some protection at C, it must be the case that $I_C < \frac{\beta-\gamma}{\beta}$. The law of motion for the multiplier is given by

$$\dot{\lambda}(t) = -\frac{c_T}{\alpha}\left[\rho+\gamma-\beta\right]+\left[\omega-2c_P\right] > 0 \text{ if } C \text{ is allowable} \tag{72}$$

and thus it follows that

$$t_3 = \frac{\dot{\lambda}(t)}{\dot{I}(t)} \tag{73}$$

$$= \frac{-\frac{c_T}{\alpha}\left[\rho+\gamma-\beta\right]+\left[\omega-2c_P\right]}{\frac{\alpha c_P}{\beta c_T}\left[\beta\left(1-\frac{\alpha c_P}{\beta c_T}\right)-\gamma\right]} > 0 \tag{74}$$

Thus, $90^0 > \theta_3 > 0$.
Angle θ_4: $\tau(t) = 0, \pi(t) = 1$. This yields the laws of motion

$$\dot{I}(t) = -\gamma I_C \tag{75}$$

$$= -\gamma\frac{\alpha c_P}{\beta c_T} < 0 \tag{76}$$

$$\dot{\lambda}(t) = \lambda_C\left[\rho+\gamma\right]+\left[\omega-c_P\right] \tag{77}$$

$$= -\frac{c_T}{\alpha}\left[\rho+\gamma\right]+\left[\omega-c_P\right] < 0 \text{ if } C \text{ is allowable} \tag{78}$$

$$t_4 = \frac{\dot{\lambda}(t)}{\dot{I}(t)} \tag{79}$$

$$= \frac{\frac{c_T}{\alpha}\left[\rho+\gamma\right]-\left[\omega-c_P\right]}{\gamma\frac{\alpha c_P}{\beta c_T}} > 0 \tag{80}$$

Thus, $90^0 > \theta_4 > 0$.

To find t_5, note that the curve with this slope satisfies the equation $\beta\lambda(t)I(t) = -c_P$ and hence at C,

$$\begin{aligned} t_5 &= \frac{d\lambda(t)}{dI(t)} && (81)\\ &= \frac{c_P}{\beta\left(I_C\right)^2} && (82)\\ &= \frac{c_P}{\beta\left(\frac{\alpha c_P}{\beta c_T}\right)^2} && (83)\\ &= \frac{\beta}{c_P}\left(\frac{c_T}{\alpha}\right)^2 > 0 && (84)\end{aligned}$$

Thus, $90^0 > \theta_5 > 0$.

Angle $\theta_1 - \theta_5$:

$$\begin{aligned} J\left(t_1 - t_5\right) &= \frac{c_T}{\alpha}\left(\rho+\gamma\right) - \left(\omega - c_P\right) - \frac{\beta}{c_P}\left(\frac{c_T}{\alpha}\right)^2\frac{c_P\alpha}{\beta c_T}(\gamma+\alpha) && (85)\\ &= \frac{c_T}{\alpha}\left(\rho+\gamma\right) - \left(\omega - c_P\right) - \frac{c_T}{\alpha}(\gamma+\alpha) && (86)\\ &= \frac{c_T}{\alpha}\rho - \left(\omega + c_T - c_P\right) < 0 \text{ if } C \text{ exists} && (87)\end{aligned}$$

where

$$J \equiv \frac{\alpha c_P}{\beta c_T}(\gamma+\alpha) > 0 \quad (88)$$

Thus $90^0 > \theta_5 - \theta_1 > 0$.

Angle $\theta_2 - \theta_5$:

$$\begin{aligned} K(t_2 - t_5) &= -\frac{c_T}{\alpha}\left[\rho+\gamma-\beta\right] + \left[\omega - 2c_P\right] - \frac{c_T}{\alpha}\left[\beta\left(1-\frac{c_P\alpha}{\beta c_T}\right) - \gamma - \alpha\right] && (89)\\ &= -\frac{c_T}{\alpha}\left[\rho-\alpha\right] + \left[\omega - c_P\right] > -\frac{c_T}{\alpha}\left[\rho-\alpha\right] + \rho\frac{c_T}{\alpha} - c_T = 0 \text{ if } C \text{ exists} && (90)\end{aligned}$$

where

$$K \equiv \frac{\alpha c_P}{\beta c_T}\left[\beta\left(1-\frac{\alpha c_P}{\beta c_T}\right) - \gamma - \alpha\right] \quad (91)$$

Thus, $180^0 > \theta_2 - \theta_5 > 0$.

Angle $\theta_3 - \theta_5$:

$$\begin{aligned} L(t_3 - t_5) &= -\frac{c_T}{\alpha}\left[\rho+\gamma-\beta\right] + \left[\omega - 2c_P\right] - \frac{c_T}{\alpha}\left[\beta\left(1-\frac{c_P\alpha}{\beta c_T}\right) - \gamma\right] && (92)\\ &= -\frac{c_T}{\alpha}\rho + \left[\omega - c_P\right] < 0 \text{ if } C \text{ exists} && (93)\end{aligned}$$

where

$$L \equiv \frac{\alpha c_P}{\beta c_T}\left[\beta\left(1-\frac{\alpha c_P}{\beta c_T}\right)-\gamma\right]>0 \tag{94}$$

Thus, $90^0 > \theta_5 - \theta_3 > 0$.

Angle $\theta_4 - \theta_5$:

$$t_4 - t_5 = \frac{\frac{c_T}{\alpha}\left[\rho+\gamma\right]-\left[\omega-c_P\right]}{\gamma\frac{\alpha c_P}{\beta c_T}} - \frac{\beta}{c_P}\left(\frac{c_T}{\alpha}\right)^2 \tag{95}$$

$$\begin{aligned} M(t_4-t_5) &= \frac{c_T}{\alpha}\left[\rho+\gamma\right]-\left[\omega-c_P\right]-\frac{c_T}{\alpha}\gamma & (96)\\ &= \frac{c_T}{\alpha}\rho-\left[\omega-c_P\right]>0 \text{ if } C \text{ exists} & (97)\end{aligned}$$

where

$$M \equiv \gamma\frac{\alpha c_P}{\beta c_T}>0 \tag{98}$$

Thus, $90^0 > \theta_4 - \theta_5 > 0$.

This establishes the inequalities we wished to show. There is therefore a clockwise rotation around C. The diagram refers to the case in which $90^0 > \theta_2$. The diagram is slightly different if $180^0 > \theta_2 > 0$, but there is still a clockwise rotation around C.

Next, suppose the interior stationary solution is C_0. Then in the region of this point, there is no prevention and the local dynamics are the same as in the treatment-only model examined by Rowthorn (2006), who showed that there is a clockwise rotation around the interior stationary solution. This concludes the first part of the proof ■

Points C and C_0 are Spiral Sources. We now prove that the rotations around the interior points C and C_0 are necessarily exploding spirals. First, consider paths $(\tau(t), \pi(t))$ that maximize the planner's problem. Then the resulting system

$$\begin{aligned} \dot{I}(t) &= I(t)\left[(1-\pi(t))\beta(1-I(t))-\gamma-\tau(t)\alpha\right] & (99)\\ \dot{\lambda}(t) &= \lambda(t)\left[\rho+\gamma+\alpha\tau(t)+\beta(2I(t)(1-\pi(t))+\pi(t)-1)\right] \\ &\quad +\left[\omega+\tau(t)c_T-\pi(t)c_P\right] & (100)\end{aligned}$$

evaluated along these paths, cannot display limit cycles. This was shown by Wagener (2003) and his argument is as follows. In $(I(t), \lambda(t))$-space, consider the vector field

$$\mathcal{F} \equiv \left(\frac{\partial H^C}{\partial\lambda(t)}, \rho\lambda(t)-\frac{\partial H^C}{\partial I(t)}\right) \tag{101}$$

Let Φ_t denote the flow mapping of the system (99)-(100). Then for some initial conditions $(I(0), \lambda(0))$, we have $\Phi_t(I(0), \lambda(0)) = (I(t), \lambda(t))$, where $(I(t), \lambda(t))$ is a solution to the system for the given initial conditions. Next, consider a set of initial conditions $\Lambda(0)$. Then Φ_t maps this set into a new set $\Lambda(t)$ as follows:

$$\Lambda(t) = \{(I(t), \lambda(t)) : (I(t), \lambda(t)) = \Phi_t(I(0), \lambda(0)) \text{ for some } (I(0), \lambda(0)) \in \Lambda(0)\} \tag{102}$$

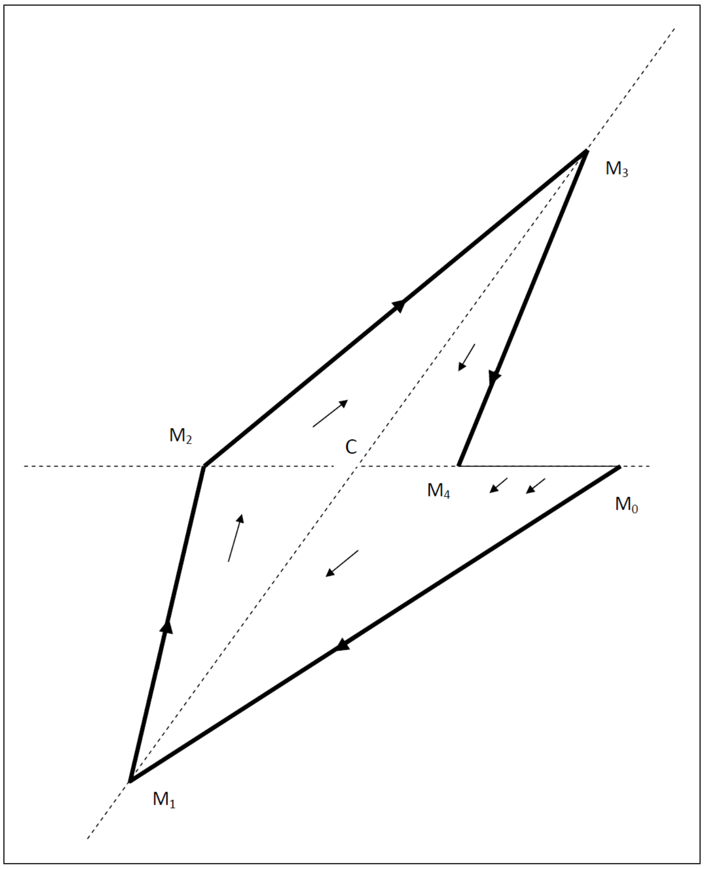

Figure 3: Rotation around point C when it is a spiral sink.

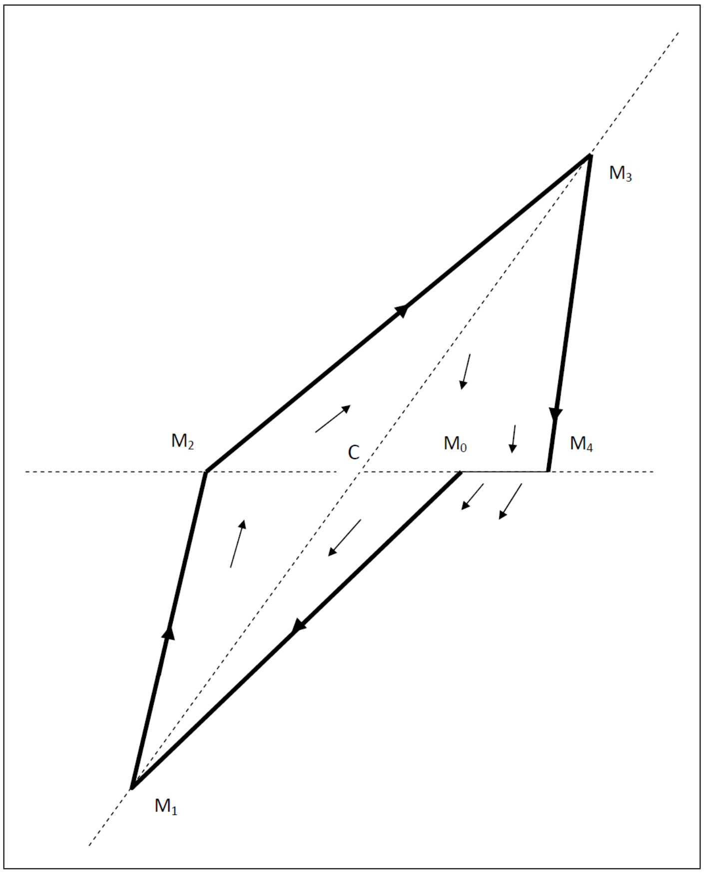

Figure 4: Rotation around point C when it is a spiral source.

The last step is to note[2] that

$$\frac{d\text{Area}\Lambda(t)}{dt}|_{t=0} = \rho\text{Area}\Lambda(0) > 0 \tag{103}$$

In other words, if we start by considering a set of initial conditions $\Lambda(0)$ with strictly positive area, then the invariant region delineated by the system must be strictly increasing over time. But this rules out limit cycles, as they would imply the existence of a bounded invariant region.

Next, we consider the possibility that the interior points are sinks. Figure 3 illustrates a segment of the trajectory around C when the point is a spiral sink. Let $\Lambda(0)$ be the closed set enclosed by the line $M_0M_1M_2M_3M_4$ together with the line segment M_4M_0. As can be seen from the figure, the initial direction of movement of every point in the set $\Lambda(0)$ is into this set, either along the boundary or into the interior. Thus

$$\frac{d\text{Area}\Lambda(t)}{dt}|_{t=0} \leq 0 \tag{104}$$

However, we have already seen that

$$\frac{d\text{Area}\Lambda(t)}{dt}|_{t=0} = \rho\text{Area}\Lambda(0) > 0 \tag{105}$$

This contradiction establishes that the point C cannot be a spiral sink. Thus, the point C must be a spiral source (with clock-wise rotation), as illustrated in Figure 4. A similar argument holds for point C_0. This concludes the second part of the proof ■

3.3. Non-Optimality of Spiraling. Next, we turn to the optimal paths starting at interior points. As discussed earlier and emphasized by the fact that the interior points are spiral sources, the Hamiltonian conditions do not pin down candidate optimal paths uniquely. It turns out that there is a simple way to determine these from a given spiraling path, as the next result shows:

Proposition 5. *A candidate optimal path starting at the prevalence levels associated with points C or C_0 is the highest or lowest monotone segment of the spiral.*

Proof: Suppose that the interior fixed point C is feasible and consider two paths which satisfy the Hamiltonian conditions and start directly above C at the points (I_C, λ_C^*) and (I_C, λ_C^{**}). Suppose $\lambda_C^{**} > \lambda_C^*$. Initially both paths satisfy the inequalities $\beta\lambda(0)I(0) > -c_P$ and $\lambda(0) > -c_T/\alpha$, and thus in each case $\pi(0) = \tau(0) = 0$. The integral along these paths satisfy the following equations:

$$\rho V^* = H^{C*} = -\omega + \lambda_C^* I_C \left[\beta(1 - I_C) - \gamma\right] \tag{106}$$

$$\rho V^{**} = H^{C*} = -\omega + \lambda_C^{**} I_C \left[\beta(1 - I_C) - \gamma\right] \tag{107}$$

Thus,

$$\rho(V^{**} - V^*) = (\lambda_C^{**} - \lambda_C^*)I_C \left[\beta(1 - I_C) - \gamma\right] > 0 \tag{108}$$

[2]See Wagener (2003) for details.

Hence, the path with the higher initial value $\lambda(t)$ is better. In the case of a spiral around the point C in $(I(t), \lambda(t))$-space, this means that it is best to choose the outermost path. This has been shown for paths that begin above the point C. A similar argument applies to paths that start below C. The rule is always choose an outermost path. These arguments also apply to the point C_0 ■

Since we know that optimal paths may form part of an explosive spiral, this result is of direct practical importance.

In our simulations, we have identified the following interesting pattern. In Regime II, where one steady state dominates the other steady state for all initial conditions, the candidate optimal path to one steady state forms part of a spiral, whereas the candidate optimal path to the other does not. In both scenarios, the non-spiraling path turns out to be the optimal one. In Regime III, i.e. the case in which there is a Skiba point, paths to both steady states form parts of nested spirals emanating from a common source. Wagener (2003) and Mäler et al. (2003) show that if there are two nested spirals that lead to distinct equilibrium points, then there exists a unique Skiba point, which is also what we find in simulations. Of course, this does not a priori mean that if there is only one spiraling path, then there is necessarily not a Skiba point. While we have not attempted a formal analysis of these observations in our setting, these seem worthwhile pursuing in future work.[3]

To conclude, we have found that the fixed points (A, B, A_0, B_0) are saddle points (if feasible), while the fixed points (C, C_0) are spiral sources.

4. Imperfect Prevention

In this appendix, we consider the effects of imperfect prevention on the steady states and dynamics of the system. Assume that for some $\delta \in [0, 1]$, the infection rate is given by

$$I(t)(1 - (1 - \delta)\pi(t))\beta \tag{109}$$

In this formulation, given the infection level $I(t)$, preventive effort $\pi(t)$ is subject to a failure rate δ. The infection rate can be brought down no further than to the level $I(t)\delta\beta$. The Hamiltonian conditions for treatment are unchanged and thus given by

$$\begin{aligned}
\tau(t) &= 0 \quad if \quad \alpha\lambda(t) > -c_T & (110)\\
\tau(t) &\in [0,1] \quad if \quad \alpha\lambda(t) = -c_T & (111)\\
\tau(t) &= 1 \quad if \quad \alpha\lambda(t) < -c_T & (112)
\end{aligned}$$

In turn, optimal prevention is given by the modified bang-bang solution

$$\begin{aligned}
\pi(t) &= 0 \quad if \quad (1-\delta)\beta\lambda(t)I(t) > -c_P & (113)\\
\pi(t) &\in [0,1] \quad if \quad (1-\delta)\beta\lambda(t)I(t) = -c_P & (114)\\
\pi(t) &= 1 \quad if \quad (1-\delta)\beta\lambda(t)I(t) < -c_P & (115)
\end{aligned}$$

[3] Note however that *when* there are two spiraling paths to the high and low infection steady states respectively, then the results of Wagener (2003) and Mäler et al. (2003) apply and there exists a unique Skiba point. This observation formally confirms a similar point made by Goldman and Lightwood (2002).

The dynamics change to

$$\begin{aligned}
\dot{I}(t) &= I(t)\left[(1-(1-\delta)\pi(t)\beta(1-I(t)) - \gamma - \tau(t)\alpha\right] && (116)\\
\dot{\lambda}(t) &= \lambda(t)\left[\rho+\gamma+\alpha\tau(t)+\beta(2I(t)-1)(1-(1-\delta)\pi(t))\right] \\
&\quad + \left[\omega+\tau(t)c_T-\pi(t)c_P\right] && (117)
\end{aligned}$$

The steady state prevalence values for points A^δ, B^δ, C^δ, A_0^δ, B_0^δ, C_0^δ, A_1^δ, B_1^δ, C_1^δ are as follows:

$$\begin{aligned}
I_{A^\delta} &\equiv \frac{\rho c_P}{\beta((1-\delta)\omega-c_P)} > I_A && (118)\\
I_{B^\delta} &\equiv \frac{\rho c_P}{\beta((1-\delta)(c_T+\omega)-c_P)} > I_B && (119)\\
I_{C^\delta} &\equiv \frac{\alpha c_P}{\beta c_T(1-\delta)} > I_C && (120)\\
I_{A_0^\delta} &\equiv \frac{\beta-\gamma}{\beta} = I_{A_0} && (121)\\
I_{B_0^\delta} &\equiv \frac{\beta-\gamma-\alpha}{\beta} = I_{B_0} && (122)\\
I_{C_0^\delta} &\equiv \frac{\alpha\omega+c_T(\beta-\gamma-\rho)}{2\beta c_T} = I_{C_0} && (123)\\
I_{A_1^\delta} &\equiv \frac{\beta\delta-\gamma}{\beta\delta} < I_{A_0^\delta} = I_{A_0} && (124)\\
I_{B_1^\delta} &\equiv \frac{\beta\delta-\gamma-\alpha}{\beta\delta} < I_{B_0^\delta} = I_{B_0} && (125)\\
I_{C_1^\delta} &\equiv \frac{\alpha\omega-\alpha c_P+c_T(\beta\delta-\gamma-\rho)}{2\beta\delta c_T} > I_{C_0^\delta} = I_{C_0} && (126)
\end{aligned}$$

A subscript 1 denotes that prevention is set at its maximal possible level. Note that compared to the levels under perfect prevention, all the relevant steady state prevalence levels are unchanged when no prevention is used, higher when an interior level of prevention is used and lower when full prevention is used.[4]

For the purpose of the non-eradication result, there are two sub-cases of particular interest. First, consider a solution with $\pi(t)=1$ and $\tau(t)=1$. A relevant steady state with these policies is feasible provided the following conditions are satisfied:

$$\begin{aligned}
\delta\beta-\gamma-\alpha &> 0 && (127)\\
\omega+c_T-c_P &> 0 && (128)\\
\alpha\left(\omega+c_T-c_P\right) &> \left(\delta\beta+\rho-\gamma-\alpha\right)c_T && (129)\\
(1-\delta)\left(\omega+c_T-c_P\right)\left(\delta\beta-\gamma-\alpha\right) &> \delta\left(\delta\beta+\rho-\gamma-\alpha\right)c_P && (130)
\end{aligned}$$

The first two conditions ensure that prevalence is interior and that the multiplier is negative. The last two conditions follow from the Hamiltonian conditions. It follows immediately from (127) that for $\delta < \underline{\delta} \equiv \gamma/\beta$, a policy with full prevention and full treatment will eradicate the disease asymptotically.

[4]The last inequality holds for $(1-\delta)\left[\alpha\omega-(\gamma+\rho)c_T\right] \geq \alpha c_P$.

Second, consider a solution with $\pi(t) = 1$ and $\tau(t) = 0$. The relevant feasibility conditions are then

$$\begin{aligned} \delta\beta - \gamma &> 0 && (131)\\ \omega - c_P &> 0 && (132)\\ \alpha\,(\omega - c_P) &< (\delta\beta + \rho - \gamma)\, c_T && (133)\\ (1-\delta)\,(\delta\beta - \gamma)\,(\omega - c_P) &> \delta\,(\delta\beta + \rho - \gamma)\, c_P && (134) \end{aligned}$$

It follows immediately from (131) that for $\delta < \overline{\delta} \equiv (\gamma+\alpha)/\beta$, a policy with full prevention but with no treatment will eradicate the disease asymptotically.

4.1. Non-Optimality of Eradication. We now confirm that under imperfect prevention, it is not optimal to eradicate the disease. Assume that $\beta > \gamma + \alpha$ and consider any path that satisfies the optimality conditions starting from $I(0) > 0$. Such a path can intersect the curve $(1-\delta)\beta\lambda(t)I(t) + c_P = 0$ at most a finite number of times and may not intersect this curve at all. There are three possibilities to consider:
(1) The path terminates at time t_0 at a fixed point $(\widehat{I}, \widehat{\lambda})$ on the curve $(1-\delta)\beta\lambda(t)I(t) + c_P = 0$. In this case, $I(t) = \widehat{I} > 0$ for $t \geq t_0$. Thus, $\lim_{t\to\infty} I(t) \neq 0$.
(2) The final segment of the path lies above the curve $\beta\lambda(t)I(t) + c_P = 0$. Hence, on this segment of the path it must be that $\pi(t) = 0$ and so

$$\dot{I}(t) = [(1-\delta)\beta(1-I(t)) - \gamma - \alpha\tau(t)]\, I(t) \geq [(1-\delta)\beta(1-I(t)) - \gamma - \alpha]\, I(t) \quad (135)$$

Since $\beta > \gamma + \alpha$, the right hand side is strictly positive for $I(t) < \frac{(1-\delta)\beta-\gamma-\alpha}{(1-\delta)\beta}$. Thus, it cannot be the case that $\lim_{t\to\infty} I(t) = 0$.
(3) The final segment of the path lies below the curve $(1-\delta)\beta\lambda(t)I(t) + c_P = 0$. Suppose that that $\lim_{t\to\infty} I(t) = 0$. Since $(1-\delta)\beta\lambda(t)I(t) < -c_P < 0$, it must be the case that $\lim_{t\to\infty} \lambda(t) = -\infty$. Thus, on the final segment of the path there must exist t_1 such that $\alpha\lambda(t) < -c_T$ for all $t \geq t_1$. This implies that $\tau(t) = 1$ for $t \geq t_1$. Since $\beta\lambda(t)I(t) < -c_P$ on the final segment, it must also be the case that $\pi(t) = 1$ for $t \geq t_1$. Hence over this range, we have that

$$\begin{aligned} \dot{I}(t) &= I(t)\,[\beta\delta(1-I(t)) - \gamma - \alpha] && (136)\\ \dot{\lambda}(t) &= \lambda(t)\,[\rho + \gamma + \alpha] + [\omega + c_T - c_P] && (137)\\ H^C(t) &= -\,[\omega + c_T]\, I(t) - c_P\,[1 - I(t)] + \lambda(t)\dot{I}(t) && (138) \end{aligned}$$

On the final segment of the path, the behavior of $I(t)$ is determined by (136). This equation will yield $\lim_{t\to\infty} I(t) = 0$ if and only if $-\beta\delta + \gamma + \alpha > 0$. Assume that this is the case and define

$$\begin{aligned} b &\equiv -\beta\delta + \gamma + \alpha > 0\\ c &\equiv \beta\delta \geq 0 \end{aligned}$$

Then over the range we are concerned with we have that

$$\dot{I}(t) = -I(t)\,[b + cI(t)]$$

It can be shown that

$$
\begin{aligned}
I(t) &= \frac{b}{\left(c+\frac{b}{I(t_1)}\right)e^{b(t-t_1)}-c} \\
\dot{I}(t) &= \frac{-b^2\left[c+\frac{b}{I(t_1)}\right]e^{b(t-t_1)}}{\left(\left[c+\frac{b}{I(t_1)}\right]e^{b(t-t_1)}-c\right)^2} \\
&= \frac{-b^2\left[c+\frac{b}{I(t_1)}\right]e^{-b(t-t_1)}}{\left(\left[c+\frac{b}{I(t_1)}\right]-ce^{-b(t-t_1)}\right)^2}
\end{aligned}
$$

Since $b > 0$, the right hand side explodes and thus the path for $I(t)$ converges to zero. However, as the following demonstration shows, it is not an optimal path. Solving equation (137) yields

$$\lambda(t) = -\frac{[\omega+c_T-c_P]}{[\rho+\gamma+\alpha]} + e^{(\rho+\gamma+\alpha)(t-t_1)}\left(\lambda(t_1)+\frac{[\omega+c_T-c_P]}{[\rho+\gamma+\alpha]}\right) \tag{139}$$

Since $\lim_{t\to\infty}\lambda(t) = -\infty$, it must be the case that

$$\lambda(t_1)+\frac{[\omega+c_T-c_P]}{[\rho+\gamma+\alpha]} < 0 \tag{140}$$

Thus, noting that $I(t)\in[0,1]$ and $b>0$, we find that

$$
\begin{aligned}
\lim_{t\to\infty}\left[e^{-\rho t}H^C(t)\right] &= \lim_{t\to\infty}\left[e^{-\rho t}\left(-[\omega+c_T]I(t)+c_P[1-I(t)]+\lambda(t)\dot{I}(t)\right)\right] \\
&= \lim_{t\to\infty}\left[e^{-\rho t}\lambda(t)\dot{I}(t)\right] \\
&= \lim_{t\to\infty}\left[e^{-\rho t}e^{(\rho+\gamma+\alpha)(t-t_1)}\left(\lambda(t_1)+\frac{[\omega+c_T-c_P]}{[\rho+\gamma+\alpha]}\right)\frac{-b^2\left[c+\frac{b}{I(t_1)}\right]e^{-b(t-t_1)}}{\left(\left[c+\frac{b}{I(t_1)}\right]-ce^{-b(t-t_1)}\right)^2}\right] \\
&= \lim_{t\to\infty}\left[e^{-\rho t}e^{(\rho+\gamma+\alpha)(t-t_1)}\left(\lambda(t_1)+\frac{[\omega+c_T-c_P]}{[\rho+\gamma+\alpha]}\right)\frac{-b^2\left[c+\frac{b}{I(t_1)}\right]e^{(\delta\beta-\gamma-\alpha)(t-t_1)}}{\left(\left[c+\frac{b}{I(t_1)}\right]-ce^{-b(t-t_1)}\right)^2}\right] \\
&= \lim_{t\to\infty}\left[-\left(\lambda(t_1)+\frac{[\omega+c_T-c_P]}{[\rho+\gamma+\alpha]}\right)\right]\frac{b^2e^{\delta\beta(t-t_1)}}{\left(c+\frac{b}{I(t_1)}\right)} > 0
\end{aligned} \tag{141}
$$

The contradicts the requirement that $\lim_{t\to\infty}\left[e^{-\rho t}H^C(t)\right] = 0$. Thus, there is no optimal path for which $\lim_{t\to\infty}I(t)=0$.

5. SUBSTITUTES, COMPLEMENTS AND SPEEDS OF CONVERGENCE

In a static model, a common definition of complementarities is that an increase in the level of one instrument increases the marginal rate of return on the other instrument. An important question in the present context is whether prevention and treatment display a similar property. For non-linear multiple-instrument optimal control problems, there are instances in which one may cleanly characterize "synergies" between the control variables, i.e. instances in which raising one control variable makes it more desirable to also raise the other (see Feichtinger, 1984). In the present model, the desirability of increasing one instrument depends on the level of the other instrument through its effects on disease prevalence. In fact, changing the level of *either* instrument influences disease prevalence, which in turn changes the desirability of further changing *both* instruments.

To see this, consider an increase in the level of prevention. Such an increase will decrease disease prevalence, thereby increasing the marginal benefits of treatment, but also decreasing the marginal benefit of prevention. Similarly, an increase in treatment will cause a decrease in disease prevalence, thereby increasing the marginal benefits of treatment, but decreasing the marginal benefits of prevention.

These interactions are simply a reflection of the insight that treatment induces a positive feedback effect, whereas prevention induces a negative feedback effect.

Almost no existing work discusses the optimal phasing of prevention and treatment. An exception is Gersovitz and Hammer (2004), who arrive at the conclusion that

> "...[optimal] subsidization [to treatment and prevention] is at equal rates because it is equally beneficial in preventing further infection to get a person out of the infected pool as to have prevented the person from getting into it in the first place [...]"

This statement seems to suggest that treatment and prevention are perfect substitutes in the steady state of their model that they consider. Our analysis shows that prevention and treatment are imperfect substitutes. Having said that, there are clearly instances in which the two instruments are used in conjunction. This stems from the fact that at some levels of disease prevalence, the strength of substitutability is low enough to render the use of both instruments optimal. This observation is intimately connected to the property of optimal paths being of the most rapid approach variety (MRAPs for short), to which we turn next.

When each policy is considered in isolation, optimal paths are known to be of this type in the prevention model but not in the treatment model (see Toxvaerd 2009a, 2010).[5] But in the present setting, this is not necessarily the case. The reason lies in the fact that the marginal benefits of treatment are decreasing in prevalence whereas the marginal benefits of prevention are increasing in prevalence. This feature of the planner's problem implies that when approaching a steady state from below and starting from very low prevalence levels, the optimal policy may involve no prevention coupled with full treatment of the (relatively few) infected individuals. As discussed earlier, this is because for low prevalence levels, the probability of reinfection is relatively modest, making treatment worthwhile, but prevention suboptimal. This implies that infection is not increasing as fast as it could. Once prevalence has increased to a level that makes further treatment undesirable, the path does become a MRAP. Similarly, when approaching a steady state

[5] More precisely, paths are always MRAPs in a setting in which recovery can only happen via treatment. If there is also spontaneous recovery, then the optimal path to the steady state from above involves no treatment, which is not an MRAP.

from above, the optimal path may involve no treatment even though there is full prevention. Again, this is because for very high prevalence levels, reinfection probabilities are so high that treatment becomes suboptimal but the marginal benefits of prevention are high enough to justify using this instrument to its fullest extent. But this means that disease prevalence does not decrease as fast as possible towards its steady state level. When (and if) prevalence has decreased to a level that makes treatment optimal, the remaining path also becomes a MRAP. In Regime III, i.e. in the case where there is a Skiba point, there is also an interior region in which optimal paths are not most rapid approach paths.

Formally, any path that spends time in areas in which $(\tau(t), \pi(t)) = (0, 1)$ or $(\tau(t), \pi(t)) = (1, 0)$, are not of the most rapid approach type. The same is true for any decreasing path in the area $(\tau(t), \pi(t)) = (1, 0)$. This implies the following observations:

Proposition 6. *(i) The optimal path to point A from the right is not a MRAP, while the optimal path from the left is potentially a MRAP. (ii) The optimal path to point B from the left is not a MRAP, while the optimal path from the right is potentially a MRAP. (iii) The optimal path to point A_0 from the right is not a MRAP, while the optimal path from the left is potentially a MRAP. (iv) Optimal paths to point B_0 are not MRAPs from either direction.*

We can further state the following:

Proposition 7. *For all paths that are potentially MRAPs, the closing segments of the paths are MRAPs.*

The previous two propositions deserve some further comments. As can be seen from Figure **??**, all paths described as "potential MRAPs" *may* involve initial segments in which the system does not approach the steady state as fast as possible. It is in this sense that they are *potentially* most rapid approach paths. Having said that, all these paths share the feature that as the system moves close enough to the steady state, the paths enter regions where they *do* approach the steady state as rapidly as possible. Thus, although some paths are not MRAPs along their entire length, their closing segments have this property.

We now turn to the behavior of the system close to the steady states. The speed of convergence towards a steady state (I, τ, π) is found via the first-order Taylor approximation[6] of the logistic growth equation around the steady state, i.e.

$$\sigma(I, \tau, \pi) \equiv -\left[(1-\pi)\beta(1-2I^*) - \alpha\tau - \gamma\right] \tag{142}$$

Because the optimal amount of preventive effort may have a discontinuity at some steady states, we need to distinguish speeds of convergence when approaching the steady state from the left and from the right respectively. We will denote by $\sigma_-(I, \tau, \pi)$ and $\sigma_+(I, \tau, \pi)$ the speeds when approaching from the left and right respectively, and $\sigma(I, \tau, \pi)$ when

[6]It is given by the equation

$$\dot{I}(t) \approx I^* \left[(1-\pi^*)\beta(1-I^*) - \gamma - \alpha\tau^*\right] + (I(t) - I^*)\left[(1-\pi^*)\beta(1-2I^*) - \alpha\tau^* - \gamma\right]$$

there is no need to distinguish the direction (because the two speeds coincide). With this notation, the speeds are given as follows:

$$\sigma_+(I_A,0,1) = \gamma \tag{143}$$

$$\sigma_-(I_A,0,0) = \gamma + \frac{c_P(\beta+2\rho)-\beta\omega}{\omega - c_P} \tag{144}$$

$$\sigma_+(I_B,1,1) = \alpha+\gamma \tag{145}$$

$$\sigma_-(I_B,1,0) = \alpha+\gamma+\frac{c_P(\beta+2\rho)-\beta(\omega+c_T)}{c_T+\omega-c_P} \tag{146}$$

$$\sigma(I_{A_0},0,0) = \beta-\gamma \tag{147}$$

$$\sigma(I_{B_0},1,0) = \beta-\gamma-\alpha \tag{148}$$

It should be emphasized that these speeds of convergence are approximations that are valid only close to the steady states in question. In particular, this means that the speed of approach of paths that contain an initial non-MRAP segment may be overstated.

Second, it is interesting to note that there is no unambiguous ranking of the speeds of convergence from the left and right to points A and B. In other words, it is not generally true that descending to points A or B with the aid of full prevention is faster than ascending to points A or B with no prevention. It depends on the cost of prevention and the relevant conditions are not implied by any of the other constraints we have maintained.[7]

6. Comparative Analysis and Welfare

The main focus of the present paper is the optimal control of infectious diseases through prevention and treatment, taking the efficiency of these interventions as given. In other words, the parameters β, α and γ are not directly controlled. Some interventions, such as the administration of antiretroviral drugs to non-infected individuals, can be interpreted as a direct change in the infectiousness of the disease (see Toxvaerd, 2010 for a discussion and a survey of that literature). It is thus also of interest to conduct comparative statics analysis with respect to these parameters and to analyze their welfare and policy implications. We shall do so in this section.

From the steady state levels listed above, the following results immediately follow:

Proposition 8. *(i) In steady states with no prevention, steady state prevalence is increasing in infectivity and decreasing in the rate of recovery. (ii) In steady states with positive prevention, steady state prevalence is decreasing in infectivity and independent of the rate of recovery.*

While infectivity is always measured by β, the rate of recovery may be γ or $(\gamma+\alpha)$, depending on steady state treatment intensity.

These results have important and surprising policy implications. They show that in the absence of prevention, the steady state comparative statics of disease prevalence with respect to infectiousness and the recovery rate, are qualitatively the same as those in the classical model. But surprisingly, when the steady state involves positive preventive

[7]Specifically, we have that $\sigma_+(I_A,0,1) > \sigma_-(I_A,0,0)$ if and only if $c_P < \frac{\beta\omega}{\beta+2\rho}$. Also, $\sigma_+(I_B,1,1) > \sigma_-(I_B,1,0)$ if and only if $c_P < \frac{\beta(\omega+c_T)}{\beta+2\rho}$.

effort, the comparative statics results are *reversed*. This is an important observation, because the decrease in infectiousness and the improvement in therapeutic technologies are an important vehicle through which medical scientists and epidemiologists seek to control epidemics. What the present results show, is that changing the basic biological parameters through direct intervention may have unexpected consequences.

To fully draw out the welfare and policy implications, we first derive two further results. First, we consider the overall welfare effects of such parameter changes and then consider the effects on steady state welfare. With these results in hand, we will be able to give a sharp characterization of the welfare tradeoff involved in changing the biological and medical parameters.

Consider the overall effects of parameter changes on discounted aggregate welfare. These are captured by changes in the optimal value function $V^*(I_0)$. We have the following results:

Proposition 9. *(i) An increase in infectiousness β decreases overall welfare. (ii) An increase in the rate of recovery $(\gamma + \alpha)$ increases overall welfare.*

Proof: From the dynamic envelope theorem, it follows that in some steady state (I, τ, π, λ), the effect of a change in a parameter x is given by[8]

$$\frac{\partial V(I_0)}{\partial x} = \int_0^\infty \frac{\partial H^C(I, \tau, \pi)}{\partial x} dt \tag{149}$$

Therefore we have that

$$\frac{\partial V(I_0)}{\partial \beta} = \int_0^\infty \lambda I(1-I)(1-\pi) dt < 0 \tag{150}$$

$$\frac{\partial V(I_0)}{\partial \alpha} = -\int_0^\infty \lambda I \tau dt \geq 0 \tag{151}$$

$$\frac{\partial V(I_0)}{\partial \gamma} = -\int_0^\infty \lambda I dt > 0 \tag{152}$$

and the result follows ■

It should be noted that the results with respect to α are strict only when the treatment level is positive (and weak if the treatment level is zero).

The comparative dynamics results with respect to α, γ, β are hardly surprising. They also follow from a simple revealed preferences argument, as noted in Toxvaerd (2010). Consider a decrease in β or an increase in either α or γ. Ceteris paribus, infection is now easier to control and the planner can always choose the same paths for disease prevalence and the policy instruments as before the change in parameters. Thus overall welfare cannot be lower after the decrease in infectiousness or the increase in the rate of recovery.

It turns out that the gains in overall welfare may have an unexpected source, depending on the steady state in question. To see this, we first determine the effects of parameter changes on steady state welfare. We find the following results:

[8] In this result, the Hamiltonian is first differentiated with respect to the parameter and only then is the resulting expression evaluated at the relevant steady state values. See Caputo (2005) for details.

Proposition 10. *(i) In steady states with no prevention, steady state welfare is decreasing in infectivity and increasing in the rate of recovery. (ii) In steady states with positive prevention, steady state welfare is increasing in infectivity if $\rho > \gamma + \alpha$ and increasing in the rate of recovery.*[9]

Proof: The steady state levels of welfare associated with the non-interior steady states are given as follows:

$$H^C(I_A, \tau_A, \pi_A, \lambda_A) = \frac{-c_P(\beta - \gamma + \rho)}{\beta} \tag{153}$$

$$H^C(I_B, \tau_B, \pi_B, \lambda_B) = \frac{-c_P(\beta - \gamma + \rho - \alpha)}{\beta} \tag{154}$$

$$H^C(I_{A_0}, \tau_{A_0}, \pi_{A_0}, \lambda_{A_0}) = \frac{-\gamma\omega}{\beta} \tag{155}$$

$$H^C(I_{B_0}, \tau_{B_0}, \pi_{B_0}, \lambda_{B_0}) = \frac{(\alpha - \beta + \gamma)(c_T + \omega)}{\beta} \tag{156}$$

The results then follow from inspection ■

Again, note that the results with respect to α are strict only when the treatment level is positive (and weak if the treatment level is zero).

Taken together, these above results have interesting implications. Start from a situation in which the system is in steady state and consider an decrease in infectiousness β. Assume furthermore that this change does not cause a shift in regime, so that the set of equilibria and their optimality remains unchanged.

In steady states without prevention, i.e. (A_0, B_0), a decrease in β causes both overall welfare and steady state welfare to increase. On the other hand, the new steady state level of disease prevalence is lower, so the planner may have to expend resources on forcing down prevalence through additional treatment, until steady state is reached.[10] Since overall welfare is higher, the extra costs borne during the transition are outweighed by the increase in the resulting steady state welfare (both suitably discounted).

In steady states with positive prevention, i.e. (A, B), a decrease in β must also increase overall welfare, as we have seen. But we also know that such a decrease in infectiousness actually decreases steady state welfare. The upshot of this is that all gains in overall welfare stem from the transition to the new steady state. Indeed, since decreasing β increases steady state prevalence when prevention is positive, the planner forces prevalence up by reducing the level of preventive effort. The cost savings associated with not having any prevention during the transition to the new steady state are so large, that they outweigh the losses in steady state welfare (both suitably discounted).

To sum up, decreasing infectiousness must always improve overall welfare. But in order to reap the benefits of lower infectiousness, the planner must pay special attention to

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[9] This condition ensures the stated result (on the effects of changes in infectiousness) for steady state B. The weaker condition $\rho > \gamma$ ensures that the result holds for steady state A. We also note that the conditions that ensure that steady state welfare in steady states A and B is increasing in infectivity β are sufficient conditions for the shadow values of infection being negative in steady states A_0 and B_0 respectively.

[10] This is the case if starting at point B_0. If starting at point A_0, the decrease will happen without further costly infection control.

the steady state the system is in. In some steady states, the optimal policy response is to *reduce* prevalence through increased treatment, trading a short term increase in infection control costs for a long term increase in steady state welfare. In other steady states, the optimal policy response is conversely to *increase* prevalence through a reduction in prevention, trading short term cost savings from reduced infection control for a long term decrease in steady state welfare.

Turning to changes in the efficiency of treatment α, some interesting patterns emerge. While changing the infectiousness parameter β could have opposing effects on overall welfare and steady state welfare, changes in α never move these two welfare measures in opposite directions. In steady states (A, A_0), there is no treatment and thus both overall welfare and steady state welfare are in fact independent of α. There are therefore no tradeoffs to consider. In steady states (B, B_0), there is full treatment and therefore overall welfare and steady state welfare are (increasing) functions of α. In this case, there is no tradeoff between the short term costs and steady state welfare since the new steady states (if different) are reached without any changes in the steady state levels of the policy instruments.

To sum up, whether steady state prevalence changes as the efficiency of treatment α is varied, depends on whether there is any prevention in steady state. In contrast, whether such a change in efficiency has any impact on welfare (overall or in steady state), depends on whether there is any treatment in steady state.

The results show that the key ingredient in creating rational disinhibition (as discussed in Toxvaerd, 2010 and Gersovitz, 2010) is prevention rather than treatment, as it is the former that gives rise to the non-classical comparative statics results.

For completeness, we would also like to comment on a seemingly counter intuitive feature of steady states with positive preventive effort. Whereas the steady state welfare levels in points (A_0, B_0), in which there is no prevention, are functions of all the relevant deep parameters, the corresponding values for points (A, B) are not. In particular, steady state welfare in point A is independent of the health premium ω whereas in point B, it is independent of both the health premium ω and the treatment cost c_T.[11] The reason for this feature is that the optimal prevention level in these steady states are such that they exactly counterweight these parameters. In other words, the parameters are present in the optimal prevention levels, which in turn cancels out these parameters in the expressions for steady state disease prevalence.

7. LOCAL STABILITY OF EQUILIBRIUM STEADY STATES

In this appendix, we show that all the non-interior equilibrium steady states under decentralized decision making are locally stable and that the unique fully interior equilibrium steady state is locally unstable.

Solution A_0^*:

We assume that $I_{A_0^*} > 0$ and hence that $\beta - \gamma > 0$. In the region of A_0^*, the control variables are $\pi(t) = 0$ and $\tau(t) = 0$. Hence the laws of motion are given by

$$\dot{I}(t) = I(t)\left[\beta(1 - I(t)) - \gamma\right] \quad (157)$$

$$\dot{\mu}(t) = \mu(t)\left[\rho + \gamma + \beta I(t)\right] + \omega \quad (158)$$

[11]That point A is independent of c_T is not surprising since this steady state involves no treatment.

Let $I(t) = I_{A_0^*} + x$ and $\mu(t) = \mu_{A_0^*} + y$. Substituting in the above equations yields

$$\dot{x} = (I_{A_0^*} + x)\left[\beta(1 - (I_{A_0^*} + x)) - \gamma\right] \quad (159)$$

$$\dot{y} = \left(\mu_{A_0^*} + y\right)\left[\rho + \gamma + \beta(I_{A_0^*} + x)\right] + \omega \quad (160)$$

Linearizing these equations gives

$$\dot{x} = I_{A_0^*}\left[\beta(1 - I_{A_0^*}) - \gamma\right] + \left[\beta(1 - 2I_{A_0^*}) - \gamma\right]x \quad (161)$$

$$\dot{y} = \mu_{A_0^*}\left[\rho + \gamma + \beta I_{A_0^*}\right] + \omega + \beta\mu_{A_0^*}x + \left[\rho + \gamma + \beta I_{A_0^*}\right]y \quad (162)$$

Since point A_0^* is a steady state, $I_{A_0^*}\left[\beta(1 - I_{A_0^*}) - \gamma\right] = 0$ and $\mu_{A_0^*}\left[\rho + \gamma + \beta I_{A_0^*}\right] + \omega = 0$. Thus,

$$\dot{x} = \left[\beta(1 - 2I_{A_0^*}) - \gamma\right]x = -(\beta - \gamma)x \quad (163)$$

$$\dot{y} = \beta\mu_{A_0^*}x + \left[\rho + \gamma + \beta I_{A_0^*}\right]y = \beta\mu_{A_0^*}x + (\beta + \rho)y \quad (164)$$

and hence

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -(\beta - \gamma) & 0 \\ \beta\mu_{A_0^*} & (\beta + \rho) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (165)$$

The eigen values are $-(\beta - \gamma)$, which is negative since $I_{A_0^*} > 0$. Also, $\beta + \rho > 0$. The convergent path corresponding to $-(\beta - \gamma)$ is

$$\beta\mu_{A_0^*}x + (2\beta + \gamma + \rho)y = 0 \quad (166)$$

This has a positive slope since $\mu_{A_0^*} < 0$. The divergent path corresponding to $(\beta + \rho)$ is

$$x = 0 \quad (167)$$

This is vertical.

Solution B_0^*:

We assume that $I_{B_0^*} > 0$ and hence that $\beta - \gamma - \alpha > 0$. The derivation in this case can be obtained from the derivations for point A_0^* by replacing γ by $(\gamma + \alpha)$. This yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -(\beta - \gamma - \alpha) & 0 \\ \beta\mu_{A_0^*} & (\beta + \rho) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (168)$$

The eigen values are $-(\beta - \gamma - \alpha)$ which is negative, since $I_{B^*} > 0$. Also, $\beta + \rho > 0$. The convergent path corresponding to $-(\beta - \gamma - \alpha)$ is

$$\beta\mu_{B_0^*}x + (2\beta + \gamma + \alpha + \rho)y = 0 \quad (169)$$

This has a positive slope since $\mu_{B_0^*} < 0$.

Solution C_0^*:

The laws of motion are

$$\dot{I}(t) = I(t)\left[(1-\pi_i(t))\beta(1-I(t)) - \gamma - \tau_i(t)\alpha\right] \quad (170)$$
$$\dot{\mu}(t) = \mu(t)\left[\rho+\gamma+(1-\pi_i(t))\beta I(t) + \tau_i(t)\alpha\right] + \omega + \tau_i(t)c_T - \pi_i(t)c_P \quad (171)$$

Let $I = I_{C_0^*} + x$ and $\mu = \mu_{C_0^*} + y$. Since $\pi = 0$, the above equations can be written as follows:

$$\dot{x} = \left(I_{C_0^*} + x\right)\left[\beta(1 - I_{C_0^*} - x) - \gamma - \tau\alpha\right] \quad (172)$$
$$= I_{C_0^*}[\beta(1 - I_{C_0^*} - x) - \gamma - \tau\alpha] + x[\beta(1 - I_{C_0^*} - x) - \gamma - \tau\alpha] \quad (173)$$
$$= I_{C_0^*}[\beta(1 - I_{C_0^*}) - \gamma - \tau_{C_0^*}\alpha] - I_{C_0^*}(\tau - \tau_{C_0^*})\alpha - \beta I_{C_0^*}x + x[\beta(1 - I_{C_0^*} - x) - \gamma - \tau\alpha] \quad (174)$$
$$= I_{C_0^*}\left[\beta(1 - I_{C_0^*}) - \gamma - \tau_{C_0^*}\alpha\right] - (\tau - \tau_{C_0^*})\alpha I_{C_0^*} + x\left[\beta(1 - 2I_{C_0^*}) - \gamma - \tau\alpha\right] - \beta x^2 \quad (175)$$

Since $I_{C_0^*}\left[\beta(1 - I_{C_0^*}) - \gamma - \tau_{C_0^*}\alpha\right] = 0$, it follows that

$$\dot{x} = -(\tau - \tau_{C_0^*})\alpha I_{C_0^*} + \left[\beta(1 - 2I_{C_0^*}) - \gamma - \tau\alpha\right]x - \beta x^2 \quad (176)$$

Also,

$$\dot{y} = \left(\mu_{C_0^*} + y\right)\left[\rho + \gamma + \beta\left(I_{C_0^*} + x\right) + \tau\alpha\right] + \omega + \tau c_T \quad (177)$$
$$= \mu_{C_0^*}\left[\rho + \gamma + \beta\left(I_{C_0^*} + x\right) + \tau_{C_0^*}\alpha\right] + \mu_{C_0^*}(\tau - \tau_{C_0^*})\alpha + \omega + \tau c_T + y\left[\rho + \gamma + \beta\left(I_{C_0^*} + x\right) + \tau\alpha\right] \quad (178)$$
$$= \mu_{C_0^*}\left[\rho + \gamma + \beta I_{C_0^*} + \tau_{C_0^*}\alpha\right] + \omega + \tau_{C_0^*}c_T + \mu_{C_0^*}\beta x + \mu_{C_0^*}(\tau - \tau_{C_0^*})\alpha + (\tau - \tau_{C_0^*})c_T \quad (179)$$
$$+ y\left[\rho + \gamma + \beta I_{C_0^*} + \tau\alpha\right] + \beta xy \quad (180)$$
$$= \left\{\mu_{C_0^*}\left[\rho + \gamma + \beta I_{C_0^*} + \tau_{C_0^*}\alpha\right] + \omega + \tau_{C_0^*}c_T\right\} + \left\{\mu_{C_0^*}\alpha + c_T\right\}(\tau - \tau_{C_0^*})$$
$$+ \mu_{C_0^*}\beta x + \left[\rho + \gamma + \beta I_{C_0^*} + \tau\alpha\right]y + \beta xy \quad (181)$$

Since $\mu_{C_0^*}\left[\rho + \gamma + \beta I_{C_0^*} + \tau_{C_0^*}\alpha\right] + \omega + \tau_{C_0^*}c_T = 0$ and $\mu_{C_0^*}\alpha + c_T = 0$, it follows that

$$\dot{y} = \mu_{C_0^*}\beta x + \left[\rho + \gamma + \beta I_{C_0^*} + \tau\alpha\right]y + \beta xy \quad (182)$$

Note that $\tau = 0$ for $y > 0$ and $\tau = 1$ for $y < 0$.

Let us consider a path that starts at time $t = 0$ at $x = x_0 < 0$ and $y = 0$ and has $\tau = 0$. For such a path, the above equations can be written as

$$\dot{x} = a_0 + b_0 x - \beta x^2 \quad (183)$$
$$\dot{y} = cx + d_0 y + \beta xy \quad (184)$$

where

$$\begin{aligned} a_0 &= \tau_{C_0^*}\alpha I_{C_0^*} > 0 && (185)\\ b_0 &= \beta(1-2I_{C_0^*})-\gamma && (186)\\ c &= \mu_{C_0^*}\beta < 0 && (187)\\ d_0 &= \rho+\gamma+\beta I_{C_0^*} > 0 && (188)\end{aligned}$$

Consider an approximate solution of the form

$$\begin{aligned} x &= x_0 + e_0 t && (189)\\ y &= g_0 t + h_0 t^2 && (190)\end{aligned}$$

As required, a solution of this type yields $x(0) = x_0 < 0$ and $y(0) = 0$. Substituting in (183) and (184) yields

$$\begin{aligned} \dot{x} &= a_0 + b_0(x_0+e_0t) - \beta\,(x_0+e_0t)^2 && (191)\\ \dot{y} &= c(x_0+e_0t) + d_0\left(g_0t+h_0t^2\right) + \beta(x_0+e_0t)\left(g_0t+h_0t^2\right) && (192)\end{aligned}$$

Ignoring higher orders of x_0 and t, the above equations can be written as follows:

$$\begin{aligned} \dot{x} &= a_0 + b_0x_0 && (193)\\ \dot{y} &= cx_0 + (ce_0 + d_0g_0)\,t && (194)\end{aligned}$$

Differentiating (189) and (190) yields

$$\begin{aligned} \dot{x} &= e_0 && (195)\\ \dot{y} &= g_0 + 2h_0t && (196)\end{aligned}$$

Comparing (193) and (194) with (195) and (196), it follows that

$$\begin{aligned} e_0 &= a_0 + b_0x_0 > 0 \text{ for sufficiently small } x_0 && (197)\\ g_0 &= cx_0 > 0 && (198)\\ h_0 &= \frac{ce_0 + d_0g_0}{2} && (199)\\ &= \frac{c\,(a_0+b_0x_0) + d_0cx_0}{2} && (200)\\ &= \frac{ca_0 + c\,(b_0+d_0)\,x_0}{2} < 0 \text{ for sufficiently small } x_0 && (201)\end{aligned}$$

Equation (190) implies that the path will achieve the value $y = 0$ for a second time at $t = t_1 = -g_0/h_0 > 0$ for small x_0. When this happens equation (190) implies that the value of x is as follows:

$$x_1 = x_0 + e_0t_1 = x_0 - \frac{e_0g_0}{h_0} \text{ for sufficiently small } x_0 \qquad (202)$$

Expanding and ignoring higher orders of x_0, we get that

$$
\begin{aligned}
x_1 &\approx x_0 - \frac{2\left(a_0+b_0x_0\right)cx_0}{ca_0+c\left(b_0+d_0\right)x_0} &(203)\\
&= x_0 - \frac{2\left(a_0+b_0x_0\right)x_0}{a_0+\left(b_0+d_0\right)x_0} &(204)\\
&= -x_0\left(\frac{-a_0-\left(b_0+d_0\right)x_0+2\left(a_0+b_0x_0\right)}{a_0+\left(b_0+d_0\right)x_0}\right) &(205)\\
&\approx -x_0\left(\frac{a_0+\left(b_0-d_0\right)x_0}{a_0+\left(b_0+d_0\right)x_0}\right) &(206)\\
&\approx -x_0\left(\frac{1+\frac{b_0-d_0}{a_0}x_0}{1+\frac{b_0+d_0}{a_0}x_0}\right) &(207)\\
&\approx -x_0\left(1+\frac{b_0-d_0}{a_0}x_0\right)\left(1-\frac{b_0+d_0}{a_0}x_0\right) &(208)\\
&\approx -x_0\left(1-\frac{2d_0}{a_0}x_0\right) &(209)\\
&= -x_0+\frac{2d_0}{a_0}x_0^2 &(210)
\end{aligned}
$$

Since $a_0 > 0$ and $d_0 > 0$ it follows that

$$x_1 > -x_0 > 0 \tag{211}$$

which implies the following that we shall use later

$$-x_1 < x_0 < 0 \tag{212}$$

Let us consider the continuation of this path when $\tau = 1$. At $t = t_1$ it is the case that $x = x_1 > 0$ and $y = 0$. The laws of motion are of the form

$$
\begin{aligned}
\dot{x} &= a_1 + b_1x - \beta x^2 &(213)\\
\dot{y} &= cx + d_1y + \beta xy &(214)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= -(1-\tau_{C_0^*})\alpha I_{C_0^*} < 0 &(215)\\
b_1 &= \beta(1-2I_{C_0^*}) - \gamma - \alpha &(216)\\
c &= \mu_{C_0^*}\beta < 0 &(217)\\
d_1 &= \rho + \gamma + \beta I_{C_0^*} + \alpha > 0 &(218)
\end{aligned}
$$

Since $x(t_1) = x_1$ and $y(t_1) = 0$, the above equations have an approximate solution of the form

$$
\begin{aligned}
x &= x_1 + e_1(t-t_1) &(219)\\
y &= g_1(t-t_1) + h_1(t-t_1)^2 &(220)
\end{aligned}
$$

Following the same procedure as in the previous case, it can be shown that

$$\begin{aligned} e_1 &= a_1 + b_1 x_1 - \beta x_1^2 < 0 \text{ for sufficiently small } x_1 && (221)\\ g_1 &= c x_1 > 0 && (222)\\ h_1 &= \frac{c e_1 + d_1 g_1 + \beta g_1 x_1}{2} < 0 \text{ for sufficiently small } x_1 && (223) \end{aligned}$$

The path will achieve the value $y = 0$ for a second time when $t = t_2$ where $t_2 - t_1 = -g_1/h_1 > 0$ for small x_1. The value of x will be as follows

$$x_2 = x_1 + e_1(t_2 - t_1) = x_1 - \frac{e_1 g_1}{h_1} \tag{224}$$

Ignoring higher orders of x_1 yields

$$\begin{aligned} x_2 &\approx x_1 - \frac{2 e_1 c x_1}{c e_1 + d_1 c x_1 + \beta c x_1^2} && (225)\\ &\approx -x_1 \left(\frac{e_1 - d_1 x_1}{e_1 + d_1 x_1} \right) && (226)\\ &\approx -x_1 \left(\frac{1 - \frac{d_1}{e_1} x_1}{1 + \frac{d_1}{e_1} x_1} \right) && (227)\\ &\approx -x_1 + \frac{2 d_1}{a_1} x_1^2 && (228) \end{aligned}$$

Since $a_1 < 0$ and $d_1 > 0$, it follows that

$$x_2 < -x_1 \tag{229}$$

We have already shown that

$$-x_1 < x_0 < 0 \tag{230}$$

Hence

$$x_2 < -x_1 < x_0 < 0 \tag{231}$$

Thus, a complete rotation around C_0^* starting at $x_0 < 0$ ends up at $x_2 < 0$, which is further away from C_0^* than x_0. Hence the curve is an outward clockwise spiral.

Solution A^*:

We assume that $I_{A^*} > 0$ and hence that $\omega - c_P > 0$. Note that $\tau = 0$ for A^*. Also $\pi = 0$ for $\beta\mu I + c_P > 0$ and $\pi = 1$ for $\beta\mu I + c_P < 0$. Writing $I(t) = I_{A^*} + x$ and $\mu(t) = \mu_{A^*} + y$, it follows that $\pi = 0$ for $\beta(\mu_{A^*} + y)(I_{A^*} + x) I + c_P > 0$. Since $\beta\mu_{A^*} I_{A^*} + c_P = 0$, it follows for small x and y, that

$$\pi = 0 \text{ if } \mu_{A^*} x + I_{A^*} y > 0 \tag{232}$$

Likewise,

$$\pi = 1 \text{ if } \mu_{A^*} x + I_{A^*} y < 0 \tag{233}$$

The laws of motion are thus given by

$$\begin{aligned}\dot{I}(t) &= I(t)\left[(1-\pi(t))\beta(1-I(t))-\gamma-\tau(t)\alpha\right] && (234)\\ \dot{\mu}(t) &= \mu(t)\left[\rho+\gamma+\tau(t)\alpha+(1-\pi(t))\beta I(t)\right] \\ &\quad+\left[\omega+\tau(t)c_T-\pi(t)c_P\right] && (235)\end{aligned}$$

which can be written as

$$\begin{aligned}\dot{x} &= (I_{A^*}+x)\left[(1-\pi)\beta\left(I_{A^*}+x\right)-\gamma\right] && (236)\\ \dot{y} &= (\mu_{A^*}+y)\left[\rho+\gamma+(1-\pi)\beta\left(I_{A^*}+x\right)\right]+\left[\omega-c_P\right] && (237)\end{aligned}$$

Consider a small perturbation such that $\mu_{A^*}x+I_{A^*}y<0$. In this case $\pi=1$ and

$$\dot{x}=-\gamma I_{A^*}+o(x)<0 \tag{238}$$

Also,

$$\begin{aligned}\dot{y} &= (\mu_{A^*}+y)\left[\rho+\gamma\right]+\left[\omega-c_P\right] && (239)\\ &= \mu_{A^*}\left[\rho+\gamma\right]+\left[\omega-c_P\right]+(\rho+\gamma)\,y && (240)\\ &= \mu_{A^*}\left[\rho+\gamma\right]-\mu_{A^*}\left[\rho+\gamma\right]+(\rho+\gamma)\,y && (241)\\ &= (\rho+\gamma)\,y && (242)\end{aligned}$$

Conversely, if $\mu_{A^*}x+I_{A^*}y>0$ then $\pi=0$ and

$$\begin{aligned}\dot{x} &= (I_{A^*}+x)\left[\beta(1-(I_{A^*}+x))-\gamma\right] && (243)\\ &= I_{A^*}\left[\beta(1-I_{A^*})-\gamma\right]+o(x) && (244)\\ &= \frac{I_{A^*}}{(\omega-c_P)}\left[(\beta-\gamma)\,\omega-(\beta+\rho)c_P\right]+o(x) && (245)\end{aligned}$$

The expression for π_{A^*} can be written as follows:

$$\pi_{A^*}=1-\frac{\gamma(\omega-c_P)}{\beta\omega-(\beta+\rho+\gamma)c_P} \tag{246}$$

By assumption, $\omega-c_P>0$. Hence, to ensure that $\pi_{A^*}<1$ it must also be the case that $\beta\omega-(\beta+\rho+\gamma)c_P>0$. To ensure that $\pi_{A^*}>0$ then requires that $\beta\omega-(\beta+\rho+\gamma)c_P>\gamma(\omega-c_P)$ and hence $(\beta-\gamma)\,\omega-(\beta+\rho)c_P>0$. The latter inequality ensures that $\dot{x}>0$ for small x.

Also, we have that

$$\begin{aligned}\dot{y} &= (\mu_{A^*}+y)\left[\rho+\gamma+\beta\left(I_{A^*}+x\right)\right]+\omega && (247)\\ &= \mu_{A^*}\left[\rho+\gamma+\beta I_{A^*}\right]+\omega+\beta\mu_{A^*}x+(\rho+\gamma+\beta I_{A^*})\,y+\beta xy && (248)\\ &= \mu_{A^*}\left[\rho+\gamma\right]+\omega-c_P+\beta\mu_{A^*}x+(\rho+\gamma+\beta I_{A^*})\,y+\beta xy && (249)\\ &= \mu_{A^*}\left[\rho+\gamma\right]-\mu_{A^*}\left[\rho+\gamma\right]+\beta\mu_{A^*}x+(\rho+\gamma+\beta I_{A^*})\,y+\beta xy && (250)\\ &= \beta\mu_{A^*}x+\frac{\omega\left(\rho+\gamma\right)}{\omega-c_P}y+\beta xy && (251)\end{aligned}$$

Next, consider the *reverse* direction path that starts at A^* and has $\pi = \tau = 0$. Using (245) and (251), this can be approximated as follows:

$$\dot{x} = -a \quad (252)$$
$$\dot{y} = bx - cy \quad (253)$$

where

$$a = \frac{I_{A^*}}{(\omega - c_P)}\left[(\beta - \gamma)\,\omega - (\beta + \rho)c_P\right] > 0 \quad (254)$$
$$b = -\beta\mu_{A^*}x > 0 \quad (255)$$
$$c = \frac{\omega\,(\rho + \gamma)}{\omega - c_P} > 0 \quad (256)$$

The above equations have the approximate solution

$$x = -at \quad (257)$$
$$y = -\frac{ab}{2}t^2 + o(t^2) \quad (258)$$

The path to A^* travels along this solution in the reverse direction along the curve

$$y = -\frac{b}{2a}x^2 \quad (259)$$

This is a rising curve that flattens out to the horizontal as it approaches A^*.

In conclusion, in the region for $\beta\mu I + c_P < 0$ there is a unique path that converges horizontally to A^*. Its local equation is $y = 0$. In the region for $\beta\mu I + c_P > 0$, there is a unique path that converges from below to A^* becoming horizontal in the limit. Its local equation is $y = -\frac{b}{2a}x^2$.

Solution B^*: The derivations for point B^* follow by replacing γ by $(\gamma + \alpha)$ in the expressions for point A^* ■

8. FURTHER CHARACTERIZATION OF THE MAVERICK'S PROBLEM

Numerical example. Below, we shall prove analytically that the maverick will never best respond by choosing interior values for prevention or treatment in steady state. Momentarily taking this feature for granted, we consider the two cases where the optimal path takes the system to points A and B, respectively. For these cases, we have calculated the values of the program for an infected maverick and for a susceptible maverick at the (socially optimal) aggregate equilibrium point. We have done this for each of the four boundary value combinations (for the two control variables) and present the results in the form of a two-person game with the column player as a susceptible maverick and the row player as the same individual when infected. Time consistency of a best response amounts to a Nash equilibrium strategy in this two-person game. An infected individual makes its choice on the basis of a certain assumption about its future behavior when susceptible. However, this later behavior must be optimal for the individual's future self when uninfected.

To confirm that this is indeed the case, first consider the parameter constellation in the main body of the text, with $\alpha = 0.2$. In this case, the optimal solution for the planner yields a path that ends at the steady state $I = I_A$. The values of the program for various boundary combinations of τ and π are shown in the table below:

		Susceptible	
		$\pi = 0$	$\pi = 1$
Infected	$\tau = 0$	$-6.21, -3.10$	$-6.87, -4.50$
	$\tau = 1$	$-42.13, -21.06$	$-30.04, 4.50$

As can be seen from the payoffs, this yields the time consistent equilibrium $\tau = 0$ and $\pi = 0$. Note that this is still a Nash equilibrium for each "player" even if interior values of the control variables are allowed.

Next, consider the example but with $\alpha = 0.5$. The optimal solution for the planner now yields a path that ends at the steady state $I = I_B$. The values of the program for various boundary combinations of τ and π are shown in the table below:

		Susceptible	
		$\pi = 0$	$\pi = 1$
Infected	$\tau = 0$	$-4.84, -0.22$	$-6.87, -4.50$
	$\tau = 1$	$-16.09, -0.73$	$-19.27, -4.50$

As can be seen from the payoffs, this yields the time consistent equilibrium $\tau = 0$ and $\pi = 0$. Note that this is still a Nash equilibrium for each "player" even if interior values of the control variables are allowed.

8.1. Non-Optimality of Interior Solutions when Optimal Path Ends at $I = I_A$.

Let us examine the interior solutions for τ and π on the assumption that the optimal path ends at $I = I_A$.

The condition $\eta(t)\beta I_A + c_P = 0$. Suppose that $\eta(t)\beta I_A + c_P = 0$. Since $\lambda_A \beta I_A + c_P = 0$, it follows that

$$\eta(t) = \lambda_A = \frac{-(\omega - c_P)}{\rho} \tag{260}$$

At point A, it is the case that $\lambda_A\, \alpha + c_T > 0$. Hence

$$\eta(t)\alpha + c_T = \lambda_A\, \alpha + c_T > 0 \tag{261}$$

and thus, $\tau = 0$.

From above, it follows that

$$\begin{aligned} 0 &= \eta(t)(\rho + \gamma + \beta I_A) + \omega + \tau\,(\eta(t)\alpha + c_T) - \pi\,(\eta(t)\beta I_A + c_P) &(262)\\ &= \lambda_A\,(\rho + \gamma) + \omega + \lambda_A\,\beta I_A &(263)\\ &= \lambda_A\,(\rho + \gamma) + \omega - c_P &(264)\end{aligned}$$

This implies that

$$\lambda_A = -\frac{\omega - c_P}{\rho + \gamma} \tag{265}$$

This is generically different from the formula $\lambda_A = -(\omega - c_P)/\rho$.

The condition $\eta(t)\alpha + c_T = 0$. In this case

$$\begin{aligned} 0 &= \eta(t)(\rho+\gamma+\beta I_A)+\omega+\tau\left(\eta(t)\alpha+c_T\right)-\pi\left(\eta(t)\beta I_A+c_P\right) && (266)\\ &= \eta(t)(\rho+\gamma+\beta I_A)+\omega-\pi\left(\eta(t)\beta I_A+c_P\right) && (267) \end{aligned}$$

Since $\eta(t)\beta I_A + c_P$ cannot be zero, there are two possibilities, namely $\pi = 0$ or $\pi = 1$. If $\pi = 0$, then

$$\begin{aligned} 0 &= c_T(\rho+\gamma+\beta I_A)+\alpha\omega && (268)\\ &= c_T(\rho+\gamma+\beta I_A)+\alpha(\omega-c_P) && (269) \end{aligned}$$

and hence

$$I_A = \frac{c_T(\rho+\gamma)+\alpha(\omega-c_P)}{c_T\beta} \tag{270}$$

This generically false, since $I_A = \frac{\rho c_P}{\beta(\omega-c_P)}$.

If $\pi = 1$ and $\eta(t)\alpha + c_T = 0$, then

$$\begin{aligned} 0 &= \eta(t)(\rho+\gamma+\beta I_A)+\omega+\tau\left(\eta(t)\alpha+c_T\right)-\pi\left(\eta(t)\beta I_A+c_P\right) && (271)\\ &= \eta(t)(\rho+\gamma+\beta I_A)+\omega-\left(\eta(t)\beta I_A+c_P\right) && (272)\\ &= -\frac{c_T}{\alpha}(\rho+\gamma)+\omega-c_P && (273) \end{aligned}$$

This is generically false. In summary, the optimum solution for the maverick cannot satisfy either of the conditions $\eta(t)\alpha + c_T = 0$ and $\eta(t)\beta I_A + c_P = 0$. Hence interior solutions for τ or π cannot be optimal and the only remaining candidates for an optimum are the boundary solutions $\tau = 0, 1$ and $\pi = 0, 1$.

8.2. Non-Optimality of Interior Solutions when Optimal Path Ends at $I = I_B$. Let us examine the interior solutions for τ and π on the assumption that the optimal path ends at $I = I_B$.

The condition $\eta(t)\beta I_B + c_P = 0$. Suppose that $\eta(t)\beta I_B + c_P = 0$. Since $\lambda_B\beta I_B + c_P = 0$, it follows that

$$\eta(t) = \lambda_B = \frac{-(\omega+c_T-c_P)}{\rho} \tag{274}$$

At point B it is the case that $\lambda_B\,\alpha + c_T < 0$. Hence

$$\eta(t)\alpha + c_T = \lambda_B\,\alpha + c_T < 0 \tag{275}$$

and thus, $\tau = 1$.

From above, it follows that

$$\begin{aligned} 0 &= \eta(t)(\rho+\gamma+\beta I_B)+\omega+\tau\left(\eta(t)\alpha+c_T\right)-\pi\left(\eta(t)\beta I_B+c_P\right) && (276)\\ &= \lambda_B\left(\rho+\gamma+\beta I_B\right)+\omega+\left(\lambda_B\,\alpha+c_T\right) && (277)\\ &= \lambda_B\left(\rho+\gamma+\alpha\right)+\omega+\lambda_B\,\beta I_B+c_T && (278)\\ &= \lambda_B\left(\rho+\gamma+\alpha\right)+\omega+c_T-c_P && (279) \end{aligned}$$

This implies that

$$\lambda_B = -\frac{\omega + c_T - c_P}{\rho + \gamma + \alpha} \tag{280}$$

This is generically different from the formula $\lambda_B = -(\omega + c_T - c_P)/\rho$.

The condition $\eta(t)\alpha + c_T = 0$. In this case

$$\begin{aligned} 0 &= \eta(t)(\rho + \gamma + \beta I_B) + \omega + \tau\left(\eta(t)\alpha + c_T\right) - \pi\left(\eta(t)\beta I_B + c_P\right) & (281)\\ &= \eta(t)(\rho + \gamma + \beta I_B) + \omega - \pi\left(\eta(t)\beta I_B + c_P\right) & (282)\end{aligned}$$

Since $\eta(t)\beta I_B + c_P$ cannot be zero, there are two possibilities, namely $\pi = 0$ or $\pi = 1$. If $\pi = 0$ then

$$\begin{aligned} 0 &= c_T(\rho + \gamma + \beta I_B) + \alpha\omega & (283)\\ &= c_T(\rho + \gamma + \beta I_B) + \alpha(\omega - c_P) & (284)\end{aligned}$$

and hence

$$I_B = \frac{c_T(\rho + \gamma) + \alpha(\omega - c_P)}{c_T\beta} \tag{285}$$

This is generically false, since $I_B = \frac{\rho c_P}{\beta(c_T + \omega - c_P)}$.

If $\pi = 1$ and $\eta(t)\alpha + c_T = 0$, then

$$\begin{aligned} 0 &= \eta(t)(\rho + \gamma + \beta I_B) + \omega + \tau\left(\eta(t)\alpha + c_T\right) - \pi\left(\eta(t)\beta I_B + c_P\right) & (286)\\ &= \eta(t)(\rho + \gamma + \beta I_B) + \omega - \left(\eta(t)\beta I_B + c_P\right) & (287)\\ &= -\frac{c_T}{\alpha}(\rho + \gamma) + \omega - c_P & (288)\end{aligned}$$

This is generically false. In summary, the optimum solution for the maverick cannot satisfy either of the conditions $\eta(t)\alpha + c_T = 0$ and $\eta(t)\beta I_B + c_P = 0$. Hence interior solutions for τ or π cannot be optimal and the only remaining candidates for an optimum are the boundary solutions $\tau = 0, 1$ and $\pi = 0, 1$.

9. Non-Sufficiency of Hamiltonian Conditions in Centralized Setting

As noted in the main text, the analysis of the centralized problem is complicated by the fact that the Hamiltonian necessary conditions for optimality of paths are not sufficient conditions. In particular, neither Mangasarian's nor Arrow's sufficiency conditions hold. This stems from the convexity of the planner's current-value Hamiltonian in the state variable. To see this, recall that the planner's current-value Hamiltonian is given by

$$H^C = -I\left[\omega + \tau c_T\right] - (1 - I)\pi c_P + \lambda\left[(1 - I)(1 - \pi)\beta I - I(\gamma + \tau\alpha)\right] \tag{289}$$

The Hessian is

$$Hessian = \begin{bmatrix} \frac{\partial^2 H^C}{\partial^2 I} & \frac{\partial^2 H^C}{\partial I \partial \tau} & \frac{\partial^2 H^C}{\partial I \partial \pi} \\ \frac{\partial^2 H^C}{\partial I \partial \tau} & \frac{\partial^2 H^C}{\partial^2 \tau} & \frac{\partial^2 H^C}{\partial \tau \partial \pi} \\ \frac{\partial^2 H^C}{\partial I \partial \pi} & \frac{\partial^2 H^C}{\partial \tau \partial \pi} & \frac{\partial^2 H^C}{\partial^2 \pi} \end{bmatrix} \tag{290}$$

$$= \begin{bmatrix} -2\pi\lambda\beta & -(c_T + \alpha\lambda) & c_p + \lambda\beta(2I-1) \\ -(c_T + \alpha\lambda) & 0 & 0 \\ c_p + \lambda\beta(2I-1) & 0 & 0 \end{bmatrix} \tag{291}$$

The first order principle minors are $-2\pi\lambda\beta$, 0 and 0 respectively. The first order principle minor $-2\pi\lambda\beta$ is strictly greater than zero for $\lambda < 0$ and $\pi > 0$. The second order principle minors are 0, $-(c_T + \alpha\lambda)^2$and $-\left[c_p + \lambda\beta(2I-1)\right]^2$ respectively. At least one of these is normally strictly negative. Thus, there are combinations of (I, τ, π) which satisfy the constraints of the problem and for which the Hessian is not negative definite and hence for which H^C is not concave in these variables.

Moreover, we cannot apply Arrow's theorem in this case, since the maximized current-value Hamiltonian $\hat{H}^C(I, \lambda, t)$ is not a concave function of the state I (holding λ and t constant).

The Hamiltonian conditions are given by

$$\hat{\tau} = 0 \quad if \quad \alpha\lambda > -c_T \tag{292}$$
$$\hat{\tau} \in [0,1] \quad if \quad \alpha\lambda = -c_T \tag{293}$$
$$\hat{\tau} = 1 \quad if \quad \alpha\lambda < -c_T \tag{294}$$

$$\hat{\pi} = 0 \quad if \quad \beta\lambda I > -c_P \tag{295}$$
$$\hat{\pi} \in [0,1] \quad if \quad \beta\lambda I = -c_P \tag{296}$$
$$\hat{\pi} = 1 \quad if \quad \beta\lambda I < -c_P \tag{297}$$

The maximized current-value Hamiltonian is given by

$$\hat{H}^C = -I\left[\omega + \hat{\tau} c_T\right] - (1-I)\hat{\pi} c_P + \lambda\left[(1-I)(1-\hat{\pi})\beta I - I(\gamma + \hat{\tau}\alpha)\right] \tag{298}$$

For the Arrow sufficiency theorem to apply, the function $\hat{H}^C(I, \lambda, t)$ must be concave in I for all $I \in [0,1]$, taking λ and t as given. We shall now show that for sufficiently small values of I, the function $\hat{H}^C(I, \lambda, t)$ is not concave in I. There are three types of case to consider as follows:

Case 1. $\alpha\lambda < -c_T$ and hence $\hat{\tau} = 1$.

If $I < \frac{-c_P}{\beta\lambda}$, then $\hat{\pi} = 0$ and

$$\hat{H}^C(I, \lambda, t) = -I\left[\omega + c_T\right] + \lambda\left[(1-I)\beta I - I(\gamma + \alpha)\right] \tag{299}$$

If $I = \frac{-c_P}{\beta\lambda}$, then the coefficient of $\hat{\pi}$ is zero and

$$\hat{H}^C(I, \lambda, t) = -I\left[\omega + c_T\right] + \lambda\left[(1-I)\beta I - I(\gamma + \alpha)\right] \tag{300}$$

Case 2. $\alpha\lambda = -c_T$ and hence the coefficient of $\hat{\tau}$ is zero.
If $I < \frac{-c_P}{\beta\lambda}$, then $\hat{\pi} = 0$ and

$$\hat{H}^C(I, \lambda, t) = -I\omega + \lambda\left[(1 - I)\beta I - I\gamma\right] \tag{301}$$

If $I = \frac{-c_P}{\beta\lambda}$, then the coefficient of $\hat{\pi}$ is zero and

$$\hat{H}^C(I, \lambda, t) = -I\omega + \lambda\left[(1 - I)\beta I - I\gamma\right] \tag{302}$$

Case 3. $\alpha\lambda > -c_T$ and hence $\hat{\tau} = 0$.
If $I < \frac{-c_P}{\beta\lambda}$, then $\hat{\pi} = 0$ and

$$\hat{H}^C(I, \lambda, t) = -I\omega + \lambda\left[(1 - I)\beta I - I\gamma\right] \tag{303}$$

If $I = \frac{-c_P}{\beta\lambda}$, then the coefficient of $\hat{\pi}$ is zero and

$$\hat{H}^C(I, \lambda, t) = -I\omega + \lambda\left[(1 - I)\beta I - I\gamma\right] \tag{304}$$

Hence, whatever the given value of $\lambda(< 0)$, if $I \leq \frac{-c_P}{\beta\lambda}$ then the Hamiltonian $\hat{H}^C(I, \lambda, t)$ is not concave in I. Thus, the conditions of the Arrow theorem are not satisfied.

10. Quarantine versus Prevention

Consider the setting in which the planner can choose the fraction $q(t) \in [0, 1]$ of infected individuals that are quarantined. Quarantine costs $c_Q \geq 0$ per instant per infected individual. Quarantine reduces the contact rates between infected and susceptible individuals and hence disease incidence becomes

$$(1 - q(t))\beta I(t)(1 - I(t)) \tag{305}$$

This is virtually the same as under prevention as we have modeled it so far. The main difference appears in the cost of the intervention, which depends on which class of individuals is being targeted.

The planner's problem is given by

$$\max_{\tau(t), q(t) \in [0,1]} \int_0^\infty e^{-\rho t}\left[I(t)(\omega_I - c_T\tau(t)) + (1 - I(t))\omega_S - I(t)q(t)c_Q\right] dt \tag{306}$$

Disease prevalence evolves according to the differential equation

$$\dot{I}(t) = I(t)\left[(1 - q(t))\beta(1 - I(t)) - \gamma - \tau(t)\alpha\right] \tag{307}$$

The necessary conditions for optimality (for an interior level of prevalence) are then given by

$$c_T + \lambda(t)\alpha = 0 \tag{308}$$

$$c_Q + \beta\lambda(t)(1 - I(t)) = 0 \tag{309}$$

Note that the optimality condition for treatment is unchanged, but that the condition for optimal quarantine differs from that characterizing optimal prevention.

Last, the multiplier evolves according to the differential equation

$$\dot{\lambda}(t) = \lambda(t)\left[\rho + \gamma + \tau(t)\alpha - \beta(1 - q(t))(1 - 2I(t))\right] + \left[\omega + q(t)c_Q + \tau(t)c_T\right] \tag{310}$$

This version of our model is in fact a generalization of a model analyzed by Sethi (1978). He characterizes the optimal quarantine policy in the SIS model, but without treatment as a control instrument.